California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination
Applied Probability Theory   Fall 2020
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Do five of the following seven problems as indicated below. You do not need to simplify
algebraic expressions in your final answers unless specifically asked to do so. All problems
are worth the same number of points.

To receive full credit, make sure to **give reasons for your answers**, for example explaining
how you set up an equation or what theorems you apply, and to clearly mark your answers.

Exams are graded anonymously, so put your name only where directed and follow any in-
structions concerning identification code numbers.

Probability Distributions
The density of an exponential random variable with parameter λ is given by
$$f(x) = \lambda e^{-\lambda x} \quad \text{for } x \geq 0.$$

**SECTION 1 – Do two (2) problems from this section. If you attempt all three,
then the best two will be used for your grade.**

Fall 2020 #1

(a) A device that continuously measures and records seismic activity is placed in a re-
mote region. The time, T, to failure of this device is exponentially distributed with
mean 3 years. Since the device will not be monitored during its first two years of
service, the time to discovery of its failure is $X = \max\{T, 2\}$. Calculate $\mathbb{E}(X)$.

(b) The time, T, that a manufacturing system is out of operation has cumulative distri-
bution function
$$F(t) = \begin{cases} 1 - (2/t)^2 & t > 2 \\ 0 & \text{otherwise.} \end{cases}$$
The resulting cost to the company is $Y = T^2$. Let g be the density function for Y.
Determine $g(y)$.

Fall 2020 #2

The distribution of Y, given X, is uniform on the interval $[0, X]$. The marginal density of X is

$$f(x) = \begin{cases} 2x & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Determine the joint density of X and Y. Make sure to define the domain of the density.

(b) Determine the conditional density of X, given Y. Make sure to define the domain of the density.

(c) Compute $\mathbb{E}(X^2 + Y)$.

Fall 2020 #3

(a) A company insures homes in three cities, J, K, and L. Since sufficient distance separates the cities, it is reasonable to assume that the losses occurring in these cities are mutually independent. The moment generating functions for the loss distributions of the cities are:

$$M_J(t) = (1 - 2t)^{-3}, \quad M_K(t) = (1 - 2t)^{-2.5}, \quad M_L(t) = (1 - 2t)^{-4.5}.$$

Let X represent the combined losses from the three cities. Calculate $\mathbb{E}(X^3)$.

(b) Let Y and Z be independent random variables with common moment generating function $M(t) = \exp(t^2/2)$. Let $U = Y + Z$ and $V = Y - Z$. Determine the joint moment generating function $M(t_1, t_2)$ of U and V.

SECTION 2 – Do three (3) problems from this section. If you attempt more than three, then the best three will be used for your grade.

Fall 2020 #4

(a) A coin that comes up heads with probability p is continually flipped until the pattern HT appears. (That is, you stop flipping when the most recent flip lands tails and the one immediately preceding it lands heads.) Let X denote the number of flips made. Find an explicit expression for $\mathbb{E}(X)$.

(b) The coin is flipped until the pattern HHT appears. Let Y denote the number of flips made. Find an explicit expression for $\mathbb{E}(Y)$.

Fall 2020 #5

A Markov chain $\{X_n, n \geq 0\}$ with states $1, 2, 3$ has the transition probability matrix

$$\boldsymbol{P} = \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix} = (p_{ij}).$$

(a) Find the smallest value in $\boldsymbol{P}^2$.

(b) Find a lower bound for p_{22}^i, the probability of returning to state 2 after i steps. Use this lower bound to find a lower bound for $\sum_{i=1}^n p_{22}^i$.

(c) Show that state 2 is recurrent.

(d) Find the limiting probabilities π_j, that is, the long run proportions of time that the chain is in state j for $j = 1, 2, 3$.

Fall 2020 #6

(a) Suppose X_1 and X_2 are independent exponential random variables with parameters λ_1 and λ_2, respectively. Compute $\mathbb{P}(X_1 < X_2)$.

(b) A free COVID-19 testing site has opened at Cal State LA to serve the community. Suppose there are two stations, 1 (pre-check) and 2 (testing). Assume that the service time at station 1 is exponentially distributed with 10 min per person on average, and service time at station 2 is exponentially distributed with 20 min per person on average. A student enters at station 1. Upon completing the pre-check at station 1, the student proceeds to station 2, provided station 2 is free; otherwise, the student has to wait at station 1, blocking the entrance of other students. The student exits the site after the testing at station 2 is completed. When you arrive at the testing site, there is a single student at station 1. Compute the expected time before you exit.

Fall 2020 #7

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion process.

(a) What is the distribution of $X = 3B(2) + 2B(3)$?

(b) Let $X(t) = 2 + 3B(t)$. Compute $\mathbb{P}(X(3) > 3 \mid X(2) = 2)$. Express your answer in terms of Φ, which is the cumulative distribution function of the standard normal.

(c) Let $V(t) = e^{-t}B(e^{2t})$ for $t \geq 0$, compute $\mathbb{E}(V(2))$ and $\mathrm{Cov}(V(2), V(3))$.