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PYTHAGOREAN ALIGNMENT

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ABSTRACT

Pythagorean Alignment

By

Artin Parsanian

A new parameterization of all integer triangles through a simple diagram enables the generation of all the Primitive Pythagorean Triples (PPTs), the alignment of the major approaches regarding the treatment and production of the set of such triples, and the discovery of a whole forest of PPT trees in addition to the two currently existing ones given by Hall and Price. The mathematical bedrock for the generation of any PPT tree is a fitting branch system: a definite set of co-prime pairs that is required for input into its corresponding PPT formula to produce as a tree exactly all the PPTs. Various techniques then allow us to create and transform branch systems that formulate into the fruit, the PPTs, thus growing many forms of trees.

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CHAPTER 1

Introduction

The Pythagorean triples are among the most extensively studied mathematical objects in human history, starting from the Sumerian and continuing until the most recent mathematical literature, having a constant and central presence.

In the last few centuries a direction towards generalization of known mathematical formulae exceedingly took dominance in the mathematical community. The mainstream branches of mathematics (e.g. abstract algebra) largely emerged in order to capture, arrange, and develop the known jewels of mathematics, among them the Pythagorean Theorem, the crown, and its positive Diophantine solutions, the Pythagorean triples. Many modes of generalization and treatment of the Pythagorean Theorem $a^2 + b^2 = c^2$ and the Pythagorean triples have appeared in the literature. In fact, the Pythagorean equation is a special case of various interesting generalized formulas engulfing it.

The best way to obtain all the Pythagorean triples is to first generate all the primitive Pythagorean triples (PPT), where a, b, and c are pairwise coprime, then use the conventional scalar k to easily obtain all the triples satisfying the Pythagorean Theorem. That is, if the primitive Pythagorean triple $(a, b, c) \in \mathbb{N}$ satisfies $a^2+b^2 = c^2$ and $\gcd(a, b, c) = 1$, then for all $k \in \mathbb{N}$, (ka, kb, kc) is a Pythagorean triple, only primitive when $k = 1$. This is the most popular way of organizing the Pythagorean triples, in use since at least Dickson's time the early 1900s. For gaining meaningful mathematical (number theoretical) insight into the characteristic nature and relevance of the PPTs, only positive and distinct PPTs need be considered in their definition, that is, the PPTs up to reflection and rotation, respectively. Thus, we allow the PPTs to dwell at home, in the land of Geometry.

The motivation behind this thesis is a particular diagram depicted in Figure 2.2 and its consequences, which provides us a good view of the primitive Pythagorean triples and their formulas. The known and newly found formulas for all primitive Pythagorean triples, along with their parameters, are then aligned as in Figures 4.2 and 4.3.

CHAPTER 2

The Integer Triangle Diagram and the Parametrization of Primitive Integer Triangles

Definition 2.1. *An **integer triangle** is a positive integer triple* (a, b, c)*, where* $a + b > c$*,* $b + c > a$*, and* $a + c > b$*.*

Definition 2.2. *A **primitive integer triangle**, or PIT, is a positive integer triangle* (a, b, c) *with* $\gcd(a, b, c) = 1$*.*

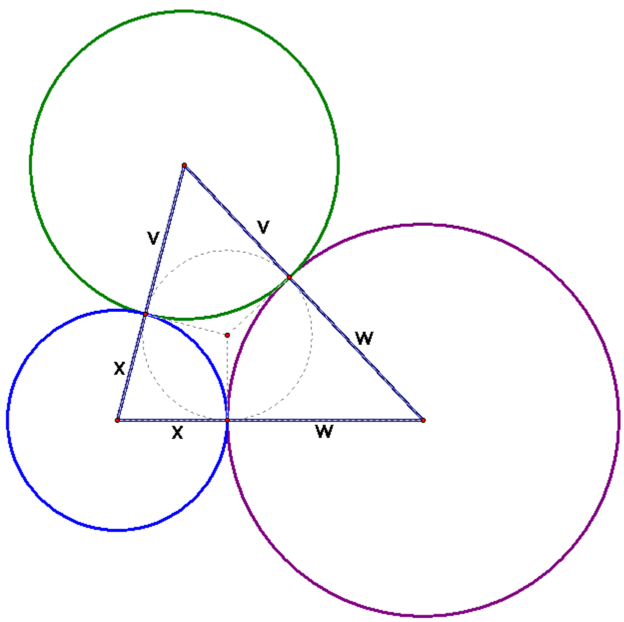

Figure 2.1: The Integer Triangle Diagram 1

As this thesis only considers integer triangles, henceforth we may use "tri-

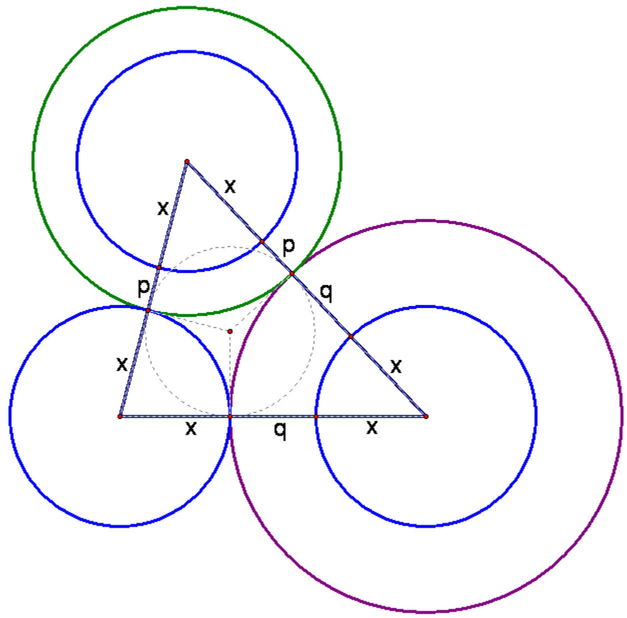

Figure 2.2: The Integer Triangle Diagram 2

angles" and "triples" interchangeably, where "triangle" is used to evoke a geometric flavor/intuition. For all integer triangles with side lengths a, b, and c, alternatively, triples (a, b, c) where a, b, and c are natural numbers and $a + b > c$, $b + c > a$, and $a + c > b$, there is an incircle with an inradius r that partitions the sides a, b, and c into segments with lengths x, v, and w as the radii of the three tangent circles apparent in the Figure 2.1. Let x be the smallest radius. Constructing circles of radius x centered at the vertices of the two larger circles, the diagram in Figure 2.2 is created. Cutting out from lengths v and w the shorter length x the remaining lengths p and q are obtained, respectively.

From Figure 2.2 emerge the next two theorems, which gives a simple categorized parametrization of all primitive integer triangles. Again, the goal is to find only the *primitive* triples because any other triple is a multiple of a primitive triple. Afterwards, with the condition $a^2 + b^2 = c^2$, attention will be restricted to the more specific Pythagorean case.

Definition 2.3. *Let s be the **semiperimeter** of an integer triangle (a, b, c) defined as $s = \dfrac{a+b+c}{2}$.*

Theorem 2.4. *Let $x, p, q \in \mathbb{R}^+$*

$v = x + p,$

$w = x + q,$

$a = x + v = 2x + p,$

$b = x + w = 2x + q,$ *and*

$c = v + w = 2x + p + q.$

Then (a, b, c) is a triangle. Moreover,

$x = \dfrac{(a+b-c)}{2},$

$v = \dfrac{(a+c-b)}{2},$

$w = \dfrac{(b+c-a)}{2},$

$p = v - x = c - b,$

$q = w - x = c - a,$ *and*

$s = x + v + w = 3x + p + q.$

Proof. We have that $a + b = (2x + p) + (2x + q) = 4x + p + q > 2x + p + q = c$, $a + c = (2x + p) + (2x + p + q) = 4x + 2p + q > 2x + q = b$, and

$c + b = (2x + p + q) + (2x + q) = 4x + p + 2q > 2x + p = a.$

So $(a, b, c) = (2x + p, 2x + q, 2x + p + q)$ is a triangle.

With substitution and simple arithmetic the rest follows.

$\square$

Nothing specific is mentioned yet about the nature of the values of x, p, q, x, v, w, and s other than they being positive real numbers. The next theorem categorizes the possible values of the parameters as defined in the above partition of the integer triples.

Theorem 2.5. ***The Primitive Integer Triangle Theorem (PITT)*** *:*

Let (a, b, c) be a primitive integer triangle. Let x, v, w, p, and $q \in \mathbb{R}^+$ such that

$v = x+p$, $w = x+q$, $a = x+v = 2x+p$, $b = x+w = 2x+q$, and $c = v+w = 2x+p+q$.

Then $x = \dfrac{(a+b-c)}{2}$, $v = \dfrac{(a+c-b)}{2}$, $w = \dfrac{(b+c-a)}{2}$,

$p = v - x = c - b$, $q = w - x = c - a$, *and* $s = x + v + w = 3x + p + q$.

Moreover,

(1) *If exactly one of a, b, or c is even, then p, q, x, v, w, and s are all in $\mathbb{N}$.*

(2) *Otherwise x, v, w and s are positive half integers. However, p, $q \in \mathbb{N}$.*

Proof. Let (a, b, c) be a primitive integer triangle. Again, using arithmetic gives that

$x = \dfrac{(a+b-c)}{2}$, $v = \dfrac{(a+c-b)}{2}$, $w = \dfrac{(b+c-a)}{2}$,

$p = v - x = c - b$, $q = w - x = c - a$, and $s = x + v + w = 3x + p + q$.

Three cases emerge from consideration of parity possibilities:

1. If exactly one of a, b, or c is even, then x, v, w, and s are all natural numbers.

2. If exactly one of a, b, or c is odd, then x, v, w, and s are positive half integers.

3. If a, b, and c are all odd, then x, v, w, and s are positive half integers.

The segments p and q are nonetheless always positive integers for all primitive integer triangles.

$\square$

CHAPTER 3

The Parametrization of the Primitive Pythagorean Triples

Definition 3.1. ***Primitive Pythagorean Triple (PPT)***

Let $(a, b, c) \in \mathbb{N}^3$*. When* (a, b, c) *is a primitive integer triangle and* $a^2 + b^2 = c^2$*,* (a, b, c) *is called a* ***primitive Pythagorean triple*** *.*

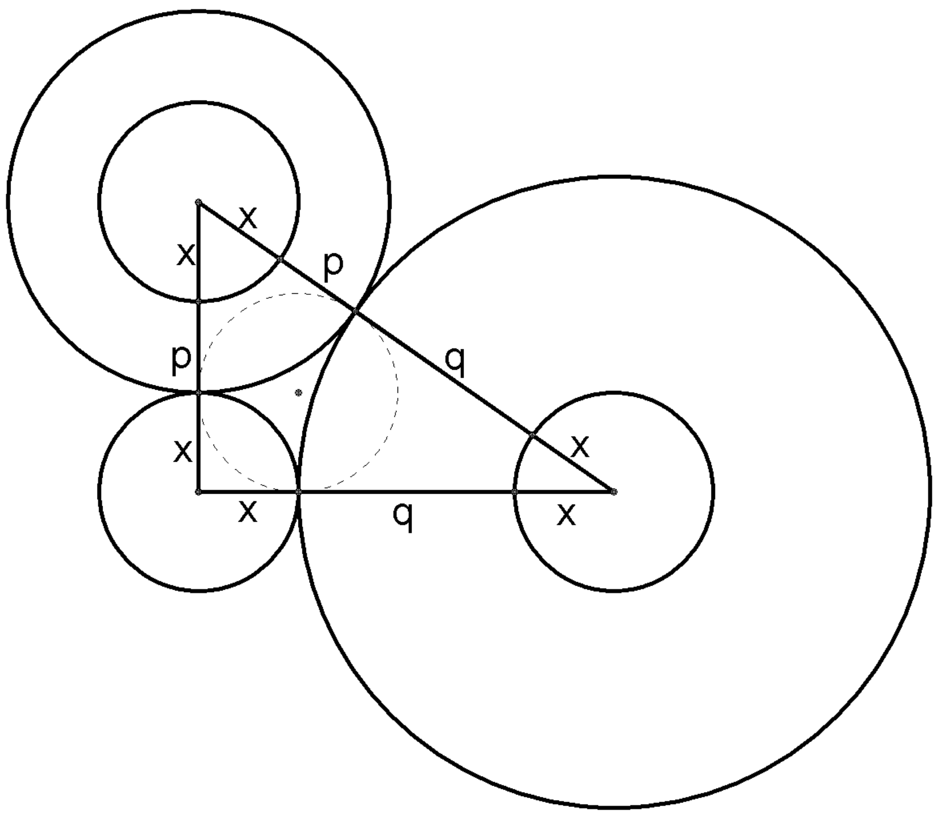

Figure 3.1: The Integer Triangle Diagram 3

Starting in this chapter, attention is focused on the primitive Pythagorean triples. As "right primitive integer triangles" or "primitive Pythagorean triangles/triples" are special cases of Theorem 2.5(1), it follows as a corollary that for all primitive Pythagorean triangles we have positive integer segments a, b, c, x, v, w, p, q, and s.

Corollary 3.2. *Let (a, b, c) be a primitive Pythagorean triple. Let x, p, q, v, w, and s be as in Theorem 2.5. Then, x, p, q, v, w, and s are all natural numbers.*

Proof. Let (a, b, c) be a primitive Pythagorean triple. Note that not all of a, b, or c are even since $\gcd(a, b, c) = 1$. Similarly, we cannot have two of a, b, or c be even since the equation $a^2 + b^2 = c^2$ implies that the third integer would have to be even contradicting the condition $\gcd(a, b, c) = 1$. Note also that we cannot have all three of a, b, and c be odd since then $a^2 + b^2 \equiv 1^2 + 1^2 \equiv 0 \mod 2$ and $c^2 \equiv 1 \mod 2$ which is a contradiction. Hence we are in case (1) of Theorem 2.5, and the result follows. □

Theorem 3.3. ***The PPT condition***

Let (a, b, c) be a primitive integer triangle. Let $x, p,$and q be as in Theorem 2.5. Then, $a^2 + b^2 = c^2$ if and only if $2x^2 = pq$.

Proof. (a, b, c) is a primitive Pythagorean triple if and only if $a^2 + b^2 = c^2$, which is true if and only if $(2x + p)^2 + (2x + q)^2 = (2x + p + q)^2$, which is true if and only if $4x^2 + 4xp + p^2 + 4x^2 + 4xq + q^2 = 4x^2 + 4x(p + q) + p^2 + 2pq + q^2$, which is true iff $2x^2 = pq$. □

Note this theorem is true for nonprimitive triples as well since the scalar k in $[k(2x + p)]^2 + [k(2x + q)]^2 = [k(2x + p + q)]^2$ would simply cancel.

Theorem 3.4. *Let* (a, b, c) *a primitive Pythagorean triple and* x, p *and* q *be as in Theorem 3.3. Then,* $\gcd(a, b) = 1$ *if and only if* $\gcd(p, q) = 1$.

Proof. By Theorem 3.3, $2x^2 = pq$. As we saw in the proof of Corollary 3.2, a and b are not both even. Suppose $2|p$ and $2|q$. Then $2|(2x + p)$ and $2|(2x + q)$. Hence $2|a$ and $2|b$. This is a contradiction. Hence p and q are not both even. Now let t be an odd prime such that $t|p$. Then $t|x$, so $t|a$. Similarly, since $b = 2x + q$, $t|q$ implies $t|b$. Conversely, let t be a prime such that $t|a$ and $t|b$. Then $t|c$ since $a^2 + b^2 = c^2$. Hence $t|(c - b)$ and $t|(c - a)$. Thus, $t|p$ and $t|q$.

□

Experimental digging into the coprime parameters p and q (i.e. their prime factorization) reveals the bedrock parameters α and β over which the primitive Pythagorean triples are constructed. As will be apparent in the theorems to come, p will be an odd square number of the form $p = \alpha^2$ and q will have the form $q = 2\beta^2$ as shown in the diagram in Figure 3.2. These parameters α and β turn out to be $m - n$ and n in the classical Greek parametrization, as will be apparent.

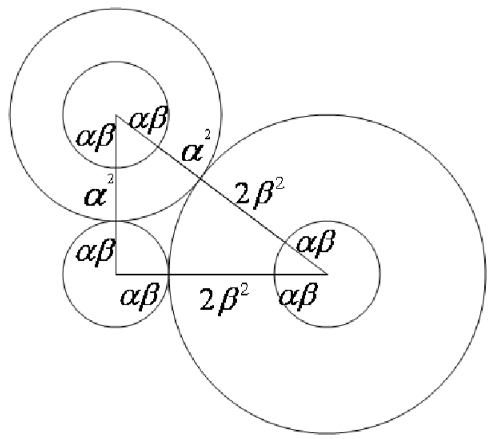

Figure 3.2: The Pythagorean Triangle Diagram with the α and β parametrization

The goal now is to define a rigorous one-to-one map from defined feeder numbers to the primitive Pythagorean triples. In addition to the seminal formula for producing the primitive integer triangle, only three primitive Pythagorean triples producing functions with their feeder ordered pairs will be considered here and encapsulated in Theorem 3.8. These three formulas will later be used to find the primitive Pythagorean triple trees and to discover the Pythagorean forest.

Below, the sets D_o, D_I, D_{II}, and D_{III} are defined, as well as their corresponding functions F_o, F_I, F_{II}, and F_{III}. Each of these sets can be used to enumerate the primitive Pythagorean triples. This is done by plugging the set into its corresponding function. This will be shown in Theorem 3.8.

Definition 3.5. *Consider the following sets with their corresponding functions:*

(1) *Let* $D_o = \{(x, p, q) \mid x, p, q \in \mathbb{N},\ 2x^2 = pq,\ \gcd(p, q) = 1\}$ *and*

$F_o : \mathbb{N}^3 \rightarrow \mathbb{N}^3$ *where* $F_o(x, p, q) = (2x + p,\ 2x + q,\ 2x + p + q)$.

(2) *Let* $D_I = \{(\alpha, \beta) \mid \alpha, \beta \in \mathbb{N},\ 2 \not| \ \alpha,\ \gcd(\alpha, \beta) = 1\}$ *and*

$F_I : \mathbb{N}^2 \rightarrow \mathbb{N}^3$ *where* $F_I(\alpha, \beta) = (2\alpha\beta + \alpha^2,\ 2\alpha\beta + 2\beta^2,\ 2\alpha\beta + \alpha^2 + 2\beta^2)$.

(3) *Let* $D_{II} = \{(m, n) \mid n, m \in \mathbb{N},\ 2|nm,\ \gcd(m, n) = 1,\ m > n\}$ *and*

$F_{II} : \mathbb{N}^2 \rightarrow \mathbb{N}^3$ *where* $F_{II}(m, n) = (m^2 - n^2,\ 2mn,\ m^2 + n^2)$.

(4) *Let* $D_{III} = \{(d, e) \mid d, e \in \mathbb{N},\ 2 \not| de,\ \gcd(d, e) = 1, d > e\}$ *and*

$F_{III} : \mathbb{N}^2 \rightarrow \mathbb{N}^3$ *where* $F_{III}(d, e) = (de, \frac{d-e}{2}, \frac{d+e}{2})$.

It is a well known fact that exactly one of the Pythagorean triples a or b in the formula $a^2 + b^2 = c^2$ is even. This propery will henceforth be part of the definition of primitive Pythagorean triples. We are concerned with the distinct triples. That is, for example, $(3, 4, 5) = (4, 3, 5)$, so it is necessary to fix either the a or the b even. It is customary to have the second number b be the even one.

Definition 3.6. *Let T be the set of primitive Pythagorean triples. That is, let* $T = \{(a, b, c) \mid a, b, c \in \mathbb{N},\ a^2 + b^2 = c^2,\ 2|b,\ \gcd(a, b) = 1\}$.

The above functions are viewed as formulas for producing T whenever their domains D_o, D_I, D_{II}, and D_{III} are either given or themselves generated. The feeder domains can be given. The classic set D_{II}, for instance, is usually represented in a two dimensional table. Otherwise they can be generated as branch systems readied for formulation into trees, as in Hall [18]. The latter one is the general approach in this thesis.

Another note, before proceeding, is that even though the homeland of our parameters p and q, or α and β, is the Integer Triangle Diagram 2 in Figure 2.2 and they are natural numbers, they and their "consecutive parameters" n and m can easily be redefined to include all integer values: positive, zero, and negative. With the inclusion of opposites and zero, this approach provides the advantage of group theoretical treatment of the Pythagorean triples, although not much insight has been demonstrated in this direction. This is to say that the parity and size restrictions that partially define the parameters in the ordered pairs (α, β), (m, n), and (d, e) are preserved even when allowing their definitions to include all integer entries. However, if we respect the right of these pairs and so the primitive Pythagorean triples to remain within their natural habitat, the land of Geometry, a more vivid and telling reflection of their many connections with the rest of mathematics (number theory) is observed. This is the preferred approach for truly understanding the essence of the primitive Pythagorean triples.

The following lemma is needed in Theorem 3.8 to show that the function F_I is a one-to-one map from the set D_I to the set T. This shows that a triple in T cannot be generated from two different ordered pairs in D_I.

Lemma 3.7. *Let (a_1, b_1, c_1) and (a_2, b_2, c_2) be primitive Pythagorean triples. Let x_1, p_1, q_1 and x_2, p_2, q_2 correspond to (a_1, b_1, c_1) and (a_2, b_2, c_2) as in Theorem 2.5. If $(a_1, b_1, c_1) = (a_2, b_2, c_2)$, then we have that $x_1 = x_2$.*

Proof. $x_1 = \dfrac{a_1 + b_1 - c_1}{2} = \dfrac{a_2 + b_2 - c_2}{2} = x_2$

$\square$

Note the converse of this lemma is false. The following is a counterexample.

Let $x_1 = x_2 = 6, p_1 = 1, q_1 = 72$, and $p_2 = 9, q_2 = 8$. Then $(a_1, b_1, c_1) = (13, 84, 85)$ and $(a_2, b_2, c_2) = (21, 20, 29)$. Here, $\alpha_1 = 1$ and $\beta_1 = 6$ whereas $\alpha_2 = 3$ and $\beta_2 = 2$.

Theorem 3.8. Primitive Pythagorean Triple Parametrization (PPTP)

Consider $D_I, D_{II}, D_{III}, F_o, F_I, F_{II}, F_{III}$, and T as in Definition 3.5.

Then, $F_o(D_o) = F_I(D_I) = F_{II}(D_{II}) = F_{III}(D_{III}) = T$.

Moreover, F_o, F_I, F_{II}, and F_{III} are one-to-one functions.

(i.e. F_J is a bijection between D_J and T for $J = o, I, II, III$).

Proof. This can be shown many ways. Figure 3.3 shows the alignment of the parameters of the ordered pairs in each set. This means that the proof of only one bijection is necessary, and the rest are equivalent through the parametric shifs shown in Figure 3.3. That is, with consistency in parity and size restrictions, setting $1 \leq \alpha = m - n = e$, $\beta = n = \dfrac{d-e}{2}$, $\alpha + \beta = m = \dfrac{d+e}{2}$, and $\alpha + 2\beta = m + n = d$.

To show that F_I is a bijection between D_I and T, we must first show that F_I is onto. Let $(a, b, c) \in T$. Thus, $a^2 + b^2 = c^2$ with $2|b$ and $\gcd(a, b) = 1$. Let x, p, q correspond to (a, b, c) as in Theorem 2.5. By Theorem 3.3, $2x^2 = pq$. Let $2|q$. Let $q = 2z$ where $z \in \mathbb{N}$. Then $x^2 = pz$. By Theorem 3.4 $\gcd(a, b) = 1$ if and only if $\gcd(p, q) = 1$. Thus $2 \not| p$. Since p and z are also relatively prime and pz is a square, we must have that both p and q are squares. That is, $p = \alpha^2$ and $z = \beta^2$ for some $\alpha, \beta \in \mathbb{N}$ with $\gcd(\alpha, \beta) = 1$. Hence $x^2 = \alpha^2\beta^2$, which gives $x = \alpha\beta$. Observe that $\gcd(\alpha, \beta) = 1$ if and only if $\gcd(p, q) = gcd(\alpha^2, 2\beta^2) = 1$. Thus $(a, b, c) = (2x + p, 2x + q, 2x + p + q) = (2\alpha\beta + \alpha^2,\ 2\alpha\beta + 2\beta^2,\ 2\alpha\beta + \alpha^2 + 2\beta^2)$ where $2 \not| \alpha$ and

$\gcd(\alpha,\beta)=1$ Therefore, $(\alpha,\beta)\in D_I$ and $F_I(\alpha,\beta)=(a,b,c)$. This proves that F_I is onto.

Now, the plan is to show that F_I is one-to-one. Let (α_1,β_1) and $(\alpha_2,\beta_2)\in D_I$. We show that if $(\alpha_1,\beta_1)\neq(\alpha_2,\beta_2)$, then $F_I(\alpha_1,\beta_1)\neq F_I(\alpha_2,\beta_2)$. We have three cases for this claim.

Case 1: Suppose that $\alpha_1\neq\alpha_2$ and $\beta_1\neq\beta_2$. Suppose the negation of the conclusion: $F_I(\alpha_1,\beta_1)=F_I(\alpha_2,\beta_2)$; that is, $(a_1,b_1,c_1)=(a_2,b_2,c_2)$. Thus $a_1=a_2$ and $2\alpha_1\beta_1+\alpha_1^2=2\alpha_2\beta_2+\alpha_2^2$. By Lemma 3.7 we have $x_1=x_2$ which gives $\alpha_1\beta_1=\alpha_2\beta_2$. The equations $\alpha_1\beta_1=\alpha_2\beta_2$ and $2\alpha_1\beta_1+\alpha_1^2=2\alpha_2\beta_2+\alpha_2^2$ give that $\alpha_1=\alpha_2$, which is a contradiciton.

Case 2: Suppose $\alpha_1\neq\alpha_2$ and $\beta_1=\beta_2$. It follows that $\alpha_1\beta_1\neq\alpha_2\beta_2$. Suppose $F_I(\alpha_1,\beta_1)=F_I(\alpha_2,\beta_2)$; that is, $(a_1,b_1,c_1)=(a_2,b_2,c_2)$. Again, since Lemma 3.7 deduces $\alpha_1\beta_1=\alpha_2\beta_2$, we have a contradiction.

Case 3: Suppose $\alpha_1=\alpha_2$ and $\beta_1\neq\beta_2$. This is similar to case 2 where the negation of the conclusion draws a contradiction.

□

The following chart is useful in picturing the alignment of the parameters of these functions.

Note, by the restrictions in their definition, the smallest possible values of the parameters in the ordered pairs of D_I, D_{II}, and D_{III} are $(1,1)$, $(1,2)$, and $(1,3)$, organized in the following table:

	odd	odd / even	even / odd	odd
I	α	β	$\alpha+\beta$	$\alpha+2\beta$
II	$m-n$	n	m	$m+n$
III	e	$\frac{d-e}{2}$	$\frac{d+e}{2}$	d

Figure 3.3: The parametric alignment of D_I, D_{II}, and D_{III}

	odd	odd / even	even / odd	odd
I	α	β		
II		n	m	
III	e			d
example	1	1	2	3

Figure 3.4: The smallest values of the parameteric pairs

CHAPTER 4

Examples and Other Pythagorean Formulas

The following is a selection of examples each with parametric value entries for (α, β), (m, n), and (d, e) followed by the primitive pythagorean triples they generate.

Notice how in this chart the Fibonacci sequential order for n and m is the reverse of the conventional ordered pair representation (m, n). Each has its virtues. The (m, n) convention allows for easy substitution into the classic formula F_{II}, but for the Fibonacci perspective, a left to right reading and a size increase, an n, m ordering is necessitated. This has apparently been a source of confusion in the past and an obstacle in the synthesis of the Pythagorean literature. It is critical to observe that there is no size restriction imposed on the α/β relation.

	(o)	(o/e)	(e/o)	(o)	a	b	c
I	α	β	$\alpha+\beta$	$\alpha+2\beta$	$2\alpha\beta+\alpha^2$	$2\alpha\beta+2\beta^2$	$2\alpha\beta+\alpha^2+2\beta^2$
II	$m-n$	n	m	$m+n$	m^2-n^2	$2mn$	m^2+n^2
III	e	$\frac{d-e}{2}$	$\frac{d+e}{2}$	d	de	$\frac{d^2-e^2}{2}$	$\frac{d^2+e^2}{2}$
ex1	1	1	2	3	3	4	5
ex2	1	2	3	5	5	12	13
ex3	1	3	4	7	7	24	25
ex4	3	1	4	5	15	8	17
ex5	3	2	5	7	21	20	29
ex6	5	1	6	7	35	12	37
ex7	5	2	7	9	45	28	53
ex8	5	3	8	11	55	48	73
ex9	5	6	11	17	85	132	157
ex10	7	1	8	9	63	16	65
ex11	7	2	9	11	77	36	85
ex12	7	3	10	13	91	60	109
ex13	7	6	13	19	133	156	205
ex14	9	7	16	23	207	224	305
ex15	11	1	12	13	143	24	145
ex16	11	4	15	19	209	120	241
ex17	13	4	17	21	273	136	305
ex18	15	8	23	31	465	368	593
ex19	17	6	23	29	493	276	565
ex20	17	13	30	43	731	780	1069
ex21	19	3	22	25	475	132	493
ex22	21	23	44	67	1407	2024	2465
ex23	23	6	29	35	805	348	877
ex24	25	1	26	27	675	52	677

Figure 4.1: Examples

4.1 Other Formulas

D_I, D_{II}, and D_{III} are not the only three distinct branch systems. There exist two more sets each with its corresponding PPT converting formula.

	odd	odd / even	even / odd	odd
I	α	β	$\alpha+\beta$	$\alpha+2\beta$
II	$m-n$	n	m	$m+n$
III	e	$\frac{d-e}{2}$	$\frac{d+e}{2}$	d
IV	$i-2j$	j	$i-j$	i
V	z	$y-z$	y	$2y-z$

Figure 4.2: The alignment of five distinct parametric pairs

	a	**b**	**c**
I: factored form	$(\alpha)(\alpha+2\beta)$	$2(\beta)(\alpha+\beta)$	$(\beta)^2+(\alpha+\beta)^2$
$F_I(D_I)$	$2\alpha\beta+\alpha^2$	$2\alpha\beta+2\beta^2$	$2\alpha\beta+\alpha^2+2\beta^2$
II: factored form	$(m-n)(m+n)$	$2(m)(n)$	$(m)^2+(n)^2$
$F_{II}(D_{II})$	m^2-n^2	$2mn$	m^2+n^2
III: factored form	$(d)(e)$	$2(\frac{d-e}{2})(\frac{d+e}{2})$	$(\frac{u+v}{2})^2+(\frac{d-e}{2})^2$
$F_{III}(D_{III})$	de	d^2-e^2	d^2+e^2
IV: factored form	$(i-2j)(i)$	$2(j)(i-j)$	$(j)^2+(i-j)^2$
$F_{IV}(D_{IV})$	i^2-2ij	$2ij-2j^2$	i^2+j^2-2ij
V: factored form	$(z)(2y-z)$	$2(y-z)(y)$	$(y-z)^2+(y)^2$
$F_V(D_V)$	$2yz-z^2$	$2y^2-2yz$	y^2+z^2-2yz

Figure 4.3: The alignment of five distinct Pythagorean formulas

These tables show the alignment of five parametric pairs and their Pythagorean formulas. As a consequence of the view these alignments afford, it is easy to see that all formulas are simple parametric shifts of one another, with the preservation of the respective parity and size conditions. The proof of these formulas is the same as that in Theorem 3.8. To the best of our knowledge, these five foundational parameteric pairs are thus far the only known distinct variable pairs that serve as feeders into their formulas to produce the set of primitive Pythagorean triples T. On close inspection, most of the Pythagorean literature on the generation of T can be reflected on these two charts.

Observe, for instance, that $(2\alpha\beta + \alpha^2,\, 2\alpha\beta + 2\beta^2\, 2\alpha\beta + \alpha^2 + 2\beta^2) = [\alpha(\alpha + 2\beta), 2\beta(\alpha + \beta), (\alpha + \beta)^2 + (\beta)^2]$. Substituting $\alpha = m - n$ and $\beta = n$ gives the classic formula $(m^2 - n^2,\, 2mn,\, m^2 + n^2)$. This is allowed because of the parity and size conditions for these parameters.

CHAPTER 5

A rigorous proof of Hall and a new binary tree

5.1 Hall's proof

In 1970, Hall gave a way to enumerate the primitive Pythagorean triples, the set T, in the form of a tree. The foundation of this tree of triples is what we call D_{II}. We now give a way to generate the set D_{II} in the form of a tree. We then apply the function F_{II} to the vertices of the tree to get the primitive Pythagorean triples. Additionally, Hall (and recently Price) found a direct way of generating the triples by using matrices, which is a combination of the tree algorithm and the classic formula F_{II} [18] [31]. Some of the trees in this article do not have this one-step-generating advantage because all the given or generated branch systems have to undergo a transformation step (e.g. sorted, trimmed) to obtain a feeder set D_J before being mapped to T.

Hall's algorithm works as follows. See Figure 5.1 for a picture of this construction.The base of the tree is the vertex $(2, 1)$. At each vertex (m, n) one applies the three formulas $(2m - n, m)$, $(2m + n, m)$, and $(m + 2n, n)$ to get the next layer of vertices. Note that given a vertex (M, N) in the tree, to go backwards in the tree one applies the formula $(N, 2N - M)$ if $N < M < 2N$, $(N, M - 2N)$ if $2N < M < 3M$, and $(M - 2N, N)$ if $3N < M$.

We now prove that the tree enumerates the elements of D_{II}. The following sequence is a rigorous and detailed version of Hall's proof for the existence of D_{II} as a branch system, after which F_{II} can simply be applied to create the set of primitive Pythagorean triples T.

Lemma 5.1. *Let $M, N \in \mathbb{N}$ such that* $\gcd(M,N) = 1$*, M and N are of opposite parity, and $M > N$. Then $M = 2N$ if and only if $(M,N) = (2,1)$.*

Proof. ($\rightarrow$) Assume $M = 2N$. If $N = 1$, then $M = 2$. If, however $N \neq 1$, then there exists a prime t that divides N, thus $t|M$, which contradicts the fact that M and N are coprime. ($\leftarrow$) If $(M,N) = (2,1)$, then $(2) = 2(1)$. $\square$

The idea in the following lemma provides a major technique for tree (branch) formation. It positions any given coprime pair (M,N) with opposite parity in one of three cases, which shows how to move to the parent of(M,N) in the Hall tree.

Lemma 5.2. *Let $M, N \in \mathbb{N}$ such that* $\gcd(M,N) = 1$*, M and N are of opposite parity, and $M > N$. Suppose $M \neq 2N$. Let $m, n \in \mathbb{N}$ be constructed as follows:*

- *If $N < M < 2N$, then $m = N$ and $n = 2N - M$.*
- *If $2N < M < 3N$, then $m = N$ and $n = M - 2N$.*
- *If $3N < M$, then $m = M - 2N$ and $n = N$.*

Then the following facts are deduced:

(1) $m > n$

(2) $\gcd(m,n) = 1$

(3) $m + n < M + N$

(4) $M \neq N$ *and* $M \neq 3N$

Proof. We break each part of this proof into cases. The cases (a), (b), and (c) correspond to the three bullet points above, respectively. In each case we make the assumptions of the respective bullet point without writing them down. We now begin the proof.

(1) (a) If $m = N$ and $n = 2N - M$, then $M = 2m - n$ and $N = m$. As $N < M < 2N$, with substitution $0 < m - n < m$, so $m > n$.

(b) If $m = N$ and $n = M - 2N$, then $M = 2m + n$ and $N = m$. As $2N < M < 3N$, substitution yields $0 < n < m$, so $m > n$.

(c) If $m = M - 2N$ and $n = N$, then $M = m + 2n$ and $N = n$. As $3N < M$, after substitution $3(n) < (m + 2n)$, so $m > n$.

(2) (a) Let t be a prime such that $t|m$. Then $t|N$. We have $\gcd(M, N) = 1$ so $t \nmid M$. Since $m = N$ and $n = 2N - M$, we have $t \nmid n$, so here $\gcd(m, n) = 1$.

(b) Let t be a prime such that $t|m$. Then $t|N$. Again with $\gcd(M, N) = 1$, $t \nmid M$. For $m = N$ and $n = M - 2N$, this gives $t \nmid n$. Here too we have that $\gcd(m, n) = 1$.

(c) Here, let t be a prime such that $t|n$. Then $t|N$. As $\gcd(M, N) = 1$, $t \nmid M$. In this case, where $m = M - 2N$ and $n = N$, it follows that $t \nmid m$.

(3) (a) In this case, we have $m + n = (N) + (2N - M) = 3N - M$. As $2N < 2M$, $3N - M < M + N$. Transitively, $m + n < M + N$.

(b) In this case, we have $m + n = (N) + (M - 2N) = M - N$. By adding $2m = 2N$ this becomes $3m + n = M + N$, which is greater than $m + n$.

(c) In this case, we have $m + n = (M - 2N) + (N) = M - N$. By adding $2n = 2N$ this becomes $m + 3n = M + N$, which is greater than $m + n$.

(4) Clearly, $M \neq N$. Suppose $M = 3N$. For any prime t dividing N, $t|M$ but

$\gcd(M, N) = 1$. The only natural number choice for N is 1, giving $M = 3$, which contradicts their opposite parity condition.

□

Theorem 5.3. *Let* $(M, N) \in \mathbb{N}^2$ *with* $\gcd(M, N) = 1$*, M and N of unlike parity, and $M > N$ (i.e. $(M, N) \in D_{II}$). (M, N) appears exactly once in the Hall branch system.*

Proof. Every (M, N) in D_{II} will fit in one of the three cases of Lemma 5.1 and Lemma 5.2. It will accordingly transform a step backwards to its generating (m, n), which seen as the new (M, N) can also be transformed in reverse, again guided by the case formula in which it fits which in turn always depends on the relation between M and N. Forwards steps are similarly guided by the corresponding case of an (m, n). A sequence of steps backwards or forwards is called a path. There is no stopping going forwards. By Lemma 5.2(3) every step backwards means a smaller coordinate sum. Any path backwards leads to a smaller endpoint. By Lemma 5.1, all backward paths ultimately lead back to where M is no longer not equal to $2N$, which is the initial vertex $(2, 1)$. Therefore, every (M, N) is on the tree as it will have a path backwards to the starting vertex. Also every (M, N) has to have a unique path backwards to $(2, 1)$ so it must appear only once on the tree.

□

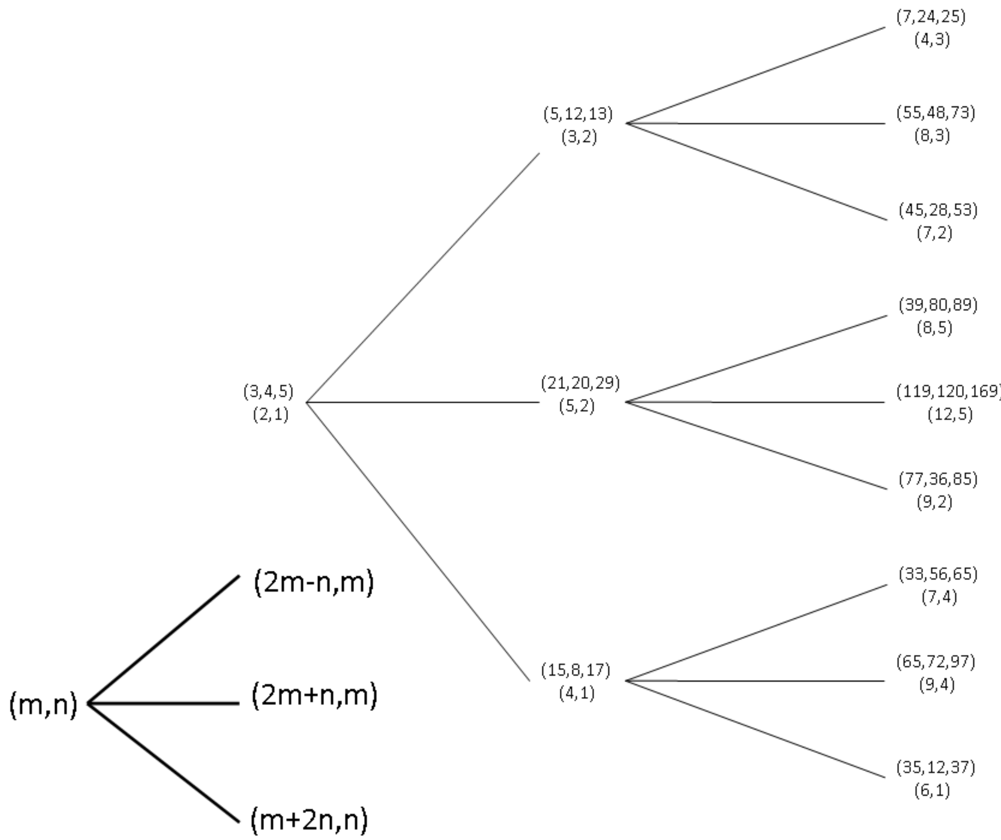

Figure 5.1: Hall Tree

5.2 A New Binary Branch System

Here we present a new mathematical tree termed **binary 1** or **B1** whose set of vertices comprise of all the reduced positive rationals, represented as coprime pairs with the first coordinate greater than the second. This set is later defined as E in Definition 6.1. The closest mathematical relative of this set is the Stern-Brocot tree, the right wing of which also generates this same set but with a different arrangement. This B1 tree can serve many other mathematical uses but we here only use it to produce primitive Pythagorean triples, and so we refer to B1 as a branch system as it serves this particular purpose. The proof that B1 contains all the elements of E is modeled on the proof of Hall's theorem/algorithm for generating a ternary tree. It turns out that $E = D_{II} \bigcup D_{III}$ and $D_{II} \bigcap D_{III} = \emptyset$. Once we know this, we will apply F_{II} to the section of B1 containing D_{II} and F_{III} to the section of B1 containing D_{III}. This will yield two copies of T. We apply similar techniques to find other trees in this thesis.

As in the production of every tree in this thesis, we introduce certain functions involving F_I, F_{II}, and F_{III}. As techniques applied to such branch system subsets of $\mathbb{N}^2$ such as **B1** the functions take as their domain from them only the necessary corresponding D_I, D_{II}, and D_{III} proper subsets to then produce the primitive Pythagorean triples on the branches in the form of trees.

The algorithm for B1, very similar to that of Hall's, works as follows. The base of the branch system is the vertex $(2, 1)$. At each vertex (g, h), one applies the two formulas $(h, h + g)$ and $(g, 2g - h)$ to get the next layer of the branch system vertices. The tree is pictured in Figure 6.1. Note that given a vertex (G, H) in the branch system, to go backwards in it one applies the formula $(H, G - H)$ if $2H < G$ and $(2H - G, H)$ if $2H > G$.

The B1 branch system can be generated with the following matrices:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix} = \begin{pmatrix} g + h \\ h \end{pmatrix}$$

$$\begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix} = \begin{pmatrix} 2g - h \\ g \end{pmatrix}$$

Lemma 5.4. *Let $G, H \in \mathbb{N}$ such that $\gcd(G, H) = 1$, and $G > H$. Then $G = 2H$ if and only if $(G, H) = (2, 1)$.*

Proof. ($\rightarrow$) Assume $G = 2H$. If $H = 1$, then $G = 2$. If, however $H \neq 1$, then there exists a prime t that divides H, thus $t|G$, which contradicts the fact that G and H are coprime. ($\leftarrow$) If $(G, H) = (2, 1)$, then $(2) = 2(1)$. $\square$

Lemma 5.5. *Let $G, H \in \mathbb{N}$ such that $\gcd(G, H) = 1$ and $G > H$. Suppose $G \neq 2H$. Let $g, h \in \mathbb{N}$ be constructed as follows:*

- *If $G > 2H$, then $g = G - H$ and $h = H$.*
- *If $G < 2H$, then $g = H$ and $h = 2H - G$.*

Then the following facts are deduced:

(1) $g > h$

(2) $\gcd(g, h) = 1$

(3) $g + h < G + H$

(4) $G \neq H$

Proof. We break each part of this proof into cases. The cases (a), (b), and (c) correspond to the three bullet points above, respectively. In each case we make the assumptions of the respective bullet point without writing them down. We now begin the proof.

(1) (a) If $g = G - H$ and $h = H$, then $G = g + h$ and $H = h$. As $g + h = G > 2H = 2h$, a subtraction of h gives $g > h$.

(b) If $g = H$ and $h = 2H - G$, then $G = 2g - h$ and $H = g$. As $2g - h = G > H = g$, arithmetic gives $g > h$. So $g > h$ in both cases.

(2) (a) Let t be a prime such that $t|h$. Since $h = H$ and $\gcd(G, H) = 1$ then $t \nmid G$. For the case where $g = G - H$ and $h = H$, we have that $t \nmid g$. Therefore, $\gcd(g, h) = 1$.

(b) Since $\gcd(G, H) = 1$, for an arbitrary prime t where $t|g, as\ g = H$, we have $t \nmid G$. For the case $g = H$ and $h = 2H - G$, it follows that $t \nmid h$. Thus, $\gcd(g, h) = 1$.

(3) (a) In this case, $g + h = G < G + H$.

(b) Since $g > h$, $2g > 2h$, and by adding $(g - h)$ to both sides $3g - h > g + h$ is obtained. As $G + H = (2g - h) + (g) = 3g - h$, transitivity provides $G + H > g + h$.

(4) $G \neq H$ as $G > H$ is given.

$\square$

Theorem 5.6. *Let* $(G, H) \in \mathbb{N}^2$ *with* $\gcd(G, H) = 1$*, and* $G > H$*. Then* (G, H) *appears exactly once in this branch system.*

Proof. The following reasoning is a near replica of Theorem 5.3.

Every (G, H) in E will fit in one of the two cases of Lemma 5.5. It will accordingly transform a step backwards to its generating (g, h), which seen as the new (G, H) can also be transformed in reverse, again guided by the case formula in which it fits which in turn always depends on the relation between G and H. Forwards steps are similarly guided by the corresponding case of an (g, h). A sequence of steps backwards or forwards is called a path. There is no stopping going forwards. By Lemma 5.5(3) every step backwards means a smaller coordinate sum. By Lemma 5.4, all backward paths ultimately lead back to where G is no longer not equal to $2H$, which is the initial vertex $(2, 1)$. Therefore, every (G, H) is on the tree as it will have a path backwards to the starting vertex. Also every (G, H) has to have a unique path backwards to $(2, 1)$ so it must appear only once on the tree.

$\square$

CHAPTER 6

The Pythagorean Forest

6.1 The Reduced Proper Rationals E and The Λ Function

Definition 6.1. *Let* $E = \{(g,h) |\ g,h \in \mathbb{N},\ \gcd(g,h) = 1,$ *and* $g > h\}$.

Definition 6.2. *Let* Λ *be a function where* $\Lambda :\ E \rightarrow T$ *and*

$$\Lambda(g,h) = \begin{cases} F_{II}(g,h) & , \quad \textit{if}\ (g,h) \in D_{II} \\ F_{III}(g,h) & , \quad \textit{if}\ (g,h) \in D_{III} \end{cases}$$

Theorem 6.3. *Let* E *and* Λ *be as in Definitions 6.1 and 6.2. Then* $\Lambda(E) = T$ *and* Λ *is a* $2-1$ *function.*

Proof. Recall Theorem 3.8 and the pairs $D_{II} = \{(m,n) \mid n,m \in \mathbb{N}, 2|nm, \gcd(m,n) = 1,\ m > n\}$ with $F_{II}(m,n) = (m^2 - n^2,\ 2mn,\ m^2 + n^2)$ and $D_{III} = \{(d,e) \mid d,e \in \mathbb{N}\ 2 / |de \gcd(d,e) = 1,\ d > e\}$ with $F_{III} = (de, \dfrac{d-e}{2}, \dfrac{d+e}{2})$. Observe that $D_{II}, D_{III} \subset E$, $E = D_{II} \bigcup D_{III}$ and $D_{II} \bigcap D_{III} = \emptyset$. Each "piece" of this piecewise-defined function, $F_I(E)$ or $F_{II}(E)$, generates exactly the set T. Thus, Λ is two-to-one and onto. $\square$

The set E is equivalent to the set of proper rationals with denominator g and numerator h. Λ as defined here is a function on each the (g,h) from E, and when these pairs are plugged into their respective formula, the output is exactly two sets of primitive Pythagorean triples. If the set E had the organizational form of a mathematical tree, what here is called a branch system, then $\Lambda(E)$ would be the fruit on that tree. Every **fruit**, say for instance the primitive pythagorean triple $(21, 20, 29)$, appears exactly *twice*. Let these types of trees be called **twin trees**. An example of this type of a tree is Figure 6.1, where Λ maps the set E (organized as $B1$) to a binary twin tree.

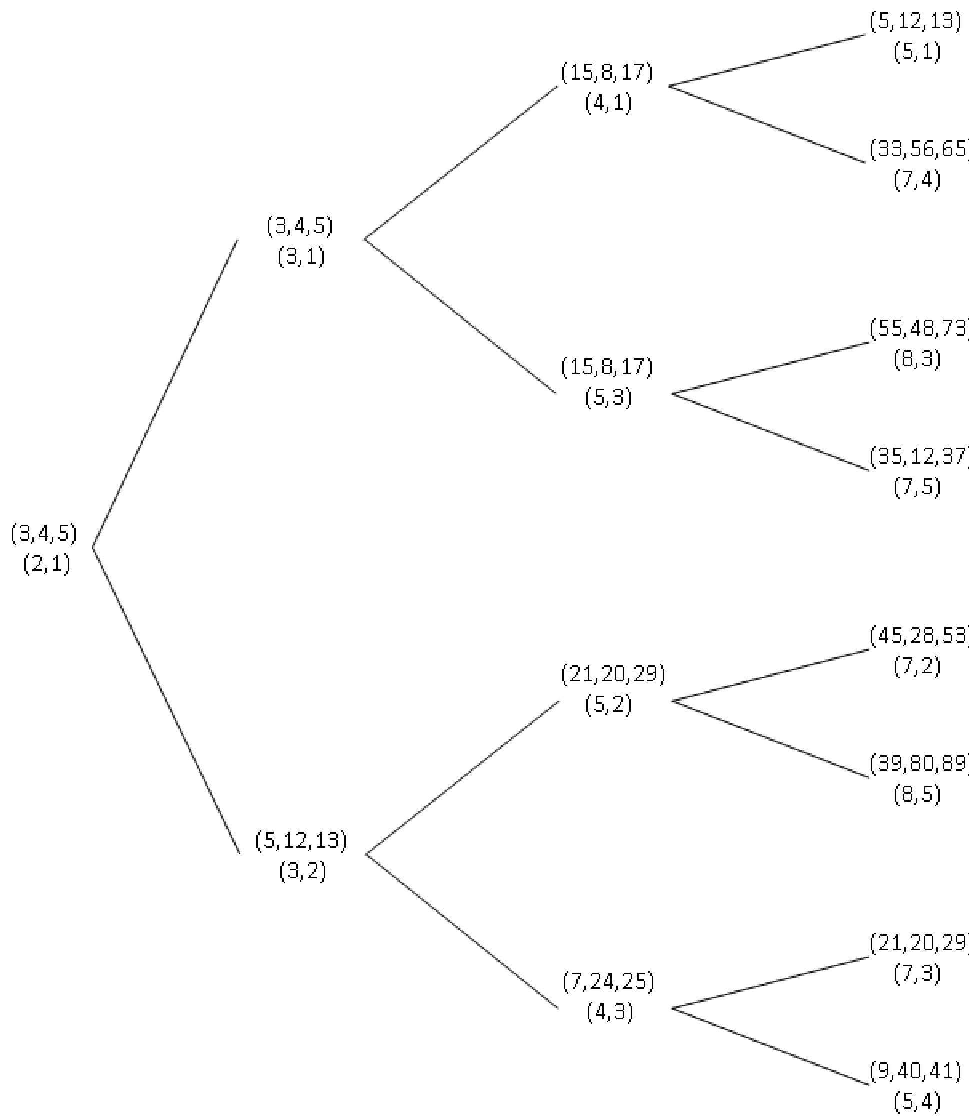

Figure 6.1: This a twin binary tree based on B1, which is an organized representation of $E = D_{II} \bigcup D_{III}$

There is another way to organize the elements of E into a tree. This can be done using the Stern-Brocot tree. The complete Stern-Brocot tree generates $\mathbb{Q}$, all the reduced rationals, and takes the form of a binary tree with coprime coordinate pairs. Its "left-wing" is symmetric to its "right-wing." The right-wing may be depicted as the set of all proper fractions, whereas the left-wing is their inverted improper fractions. Without discussing the details of how to generate this well known mathematical tree, we give its right-wing in Figure 6.1.

Definition 6.4. *Let the right wing of the Stern-Brocot tree be alternatively defined as the set* $R_{SB} \subset \mathbb{N}^2$ *where* $R_{SB} = \{(\delta, \epsilon) \mid \delta, \epsilon \in \mathbb{N},\ \gcd(\delta, \epsilon) = 1,\ \delta > \epsilon\}$.

Theorem 6.5. *Let* Λ *be as above. Then* $\Lambda(R_{SB})$ *is a twin tree. That is,* $\Lambda(R_{SB})$ *gives a tree that contains exactly two copies of* T.

Proof. The above definition of the set R_{SB} is the same as Definition 6.1 for set E. They both constitute (exactly) the set of reduced positive proper rationals, which for our purposes is alternatively viewed as the positive coprime ordered pairs with a larger "abscissa." Therefore, $\Lambda(R_{SB}) = \Lambda(E)$. By Theorem 6.3 Λ is a $2-1$ function that generates the set T. The conclusion follows. □

The chart in Figure 6.1 shows a sample of ordered pairs from $R_{SB} = E = D_{II} \bigcup D_{III}$, just a portion of the right wing of the Stern-Brocot tree of all rationals, along with their Λ-corresponding primitive Pythagorean triples. This tree looks like the B1 tree as they both constitute the same set. However, close inspection shows that it is a shffled variant.

			$(2,1)$			
			$(3,4,5)$			
	$(3,2)$				$(3,1)$	
	$(5,12,13)$				$(3,4,5)$	
$(4,3)$		$(5,3)$		$(5,2)$		$(4,1)$
$(7,24,25)$		$(15,8,17)$		$(21,20,29)$		$(15,8,17)$

Figure 6.2: The right wing of the Stern-Brocot tree as a PPT twin tree

6.2 The Set of Positive Rationals and the Ψ Function

Definition 6.6. *Let $\mathbb{Q}^+$ be the set of positive reduced rationals for convenience written in the form $\mathbb{Q}^+ = \{(g,h)|\ g,h \in \mathbb{N}$ and $\gcd(g,h) = 1\}$.*

The arrangements and representation of the elements within $\mathbb{Q}^+$ can take many forms, of most interest are those that take the form of mathematical trees refered to in this article as branch systems. The branch system, here in particular $\mathbb{Q}^+$ itself, or elsewhere in this article some subset of $\mathbb{N}^2$, serves as a domain for its corresponding function which maps it to T.

Definition 6.7. *Let $\Psi : \mathbb{Q}^+ \mapsto T \bigcup \{\emptyset\}$ such that $\forall (g,h) \in \mathbb{Q}^+$,*

$$\Psi(g,h) = \begin{cases} F_I(g,h) & , \quad \text{if } 2 \not| g \\ \emptyset & , \quad \text{if } 2|g \end{cases}$$

Theorem 6.8. *If $\mathbb{Q}^+$ is given, then $\Psi(\mathbb{Q}^+) = T \bigcup \{\emptyset\}$.*

Proof. Let $\mathbb{Q}^+$ be as in Definition 6.6. If all the ordered pairs (g,h) with an even g are mapped to $\emptyset$, then the rest of the domain will form D_I. Since $F_I(D_I) = T$, the theorem follows. $\square$

The complete Stern-Brocot tree generates $\mathbb{Q}^+$, so it may serve as a branch system for a T tree. We apply the function Ψ to the section of this branch system where the first coordinate of each vertex is odd, that is the D_I set, and map the rest of the vertices to the empty set. We call this technique **trimming**.

6.3 The Odd-Even Branch System U and the F' Function

Definition 6.9. *Let* $U = \{(g, h) \mid g, h \in \mathbb{N}, \gcd(g, h) = 1,\ 2|h\}$.

Definition 6.10. *Let* F'_I *be a function where* $F'_I \colon U \mapsto D_I$ *defined by* $F'_I(g, h) = (g, \frac{h}{2})$

Theorem 6.11. *Let* $(g, h) \in U$. *Then* $(F_I \circ F'_I)(U) = T$.

Proof. Let $(g, h) \in U$. Then, $2|h$. Since $\gcd(g, h) = 1$, $2 \not| g$. Let $h = 2z$ for $z \in \mathbb{N}$, so $z = \frac{h}{2}$. This says that $g \in \mathbb{N}$ is odd and $\frac{h}{2} \in \mathbb{N}$, which means that $(g, \frac{h}{2}) \in D_I$. Thus $F_I(g, \frac{h}{2}) \in T$. Note that $F'_I(U) = D_I$. Hence, $(F_I \circ F'_I)(U) = T$. $\square$

We now give an algorithm, call it **ternary1**, that generates a ternary branch system for the set U, whereby we can generate T via the function composition in Theorem 6.11. Again, this algorithm, as in the B1 tree can have various mathematical uses. Here though, U serves only the role of a branch system for T.

We now state an algorithm to make U into a tree. We do not provide the proof of how this algorithm yields U. The interested reader may complete the proof as it is the same as the proofs given previously in Theorem 5.3 and Theorem 5.6. The starting vertex of the tree is $(2, 1)$ and the ternary branching formulas forward for a vertex (g, h) are $(g + h, h)$ if $h < g$, $(h + g, h + 2g)$ if $h < g < 2h$, and $(g, h + 2g)$ if $2h < g$. The tree itself is pictured in Figure 6.3. After plugging the tree from Figure 6.3 into the function composition from Theorem 6.11, one gets Figure 6.4.

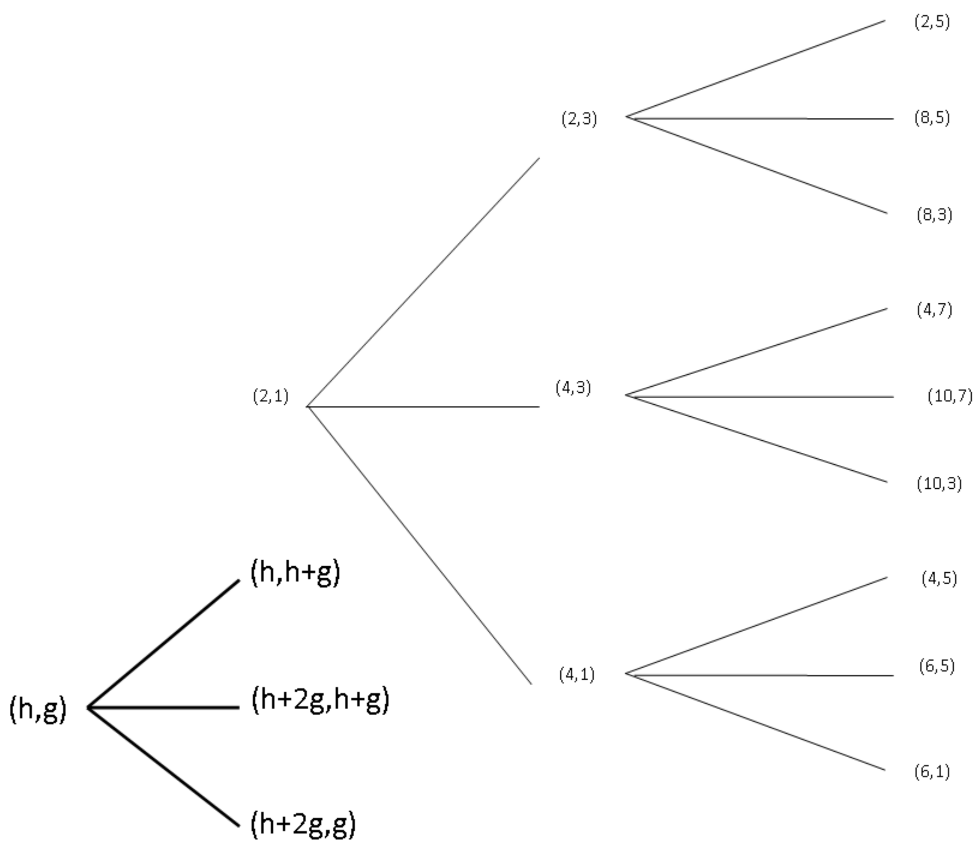

Figure 6.3: Rotated Ternary 1 (odd,even) branch system with algorithm.

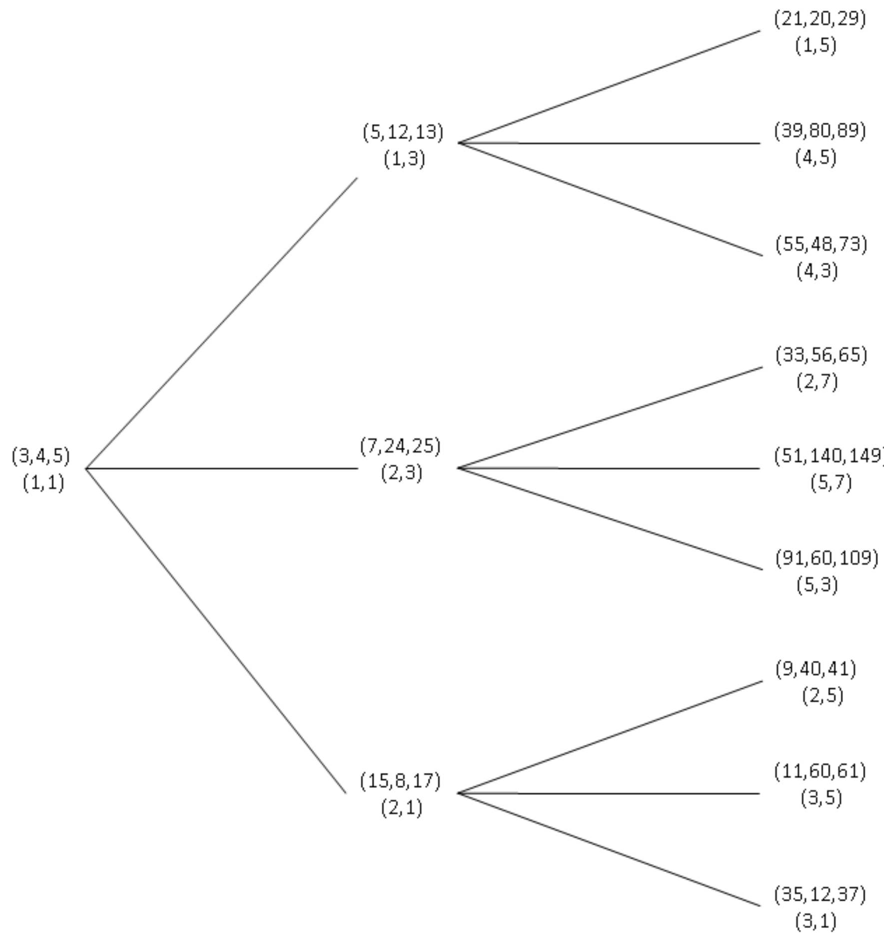

Figure 6.4: Ternary 1 (even, odd) branch system with even/2. D_I is made and F_I applied to generate T

CHAPTER 7

Conclusion

Essentially this thesis provides a foundational perspective on Pythagorean triples and their generation. This perspective was enabled by the Integer Triangle Diagrams in chapter three and the consequent alignment of the Pythagorean formulas. We now know the exact conditions under which primitive Pythagorean triples are created. Additionally, we have two new $\mathbb{N}^2$ subsets generated as a binary tree (B1) and a ternary tree (T1). With the knowledge of just the handful of functions discussed here, it is possible to create many more trees. There exist other ways of obtaining and organizing the sets E, $\mathbb{Q}^+$, or U, and other sets that may serve as branch systems. The functions such as Λ and Ψ are not the only techniques with which the set T is produced. In the capacity of this thesis we were able to represent this small yet varied sample of trees, but the main information presented in part as Theorem 3.8 is what is really necessary to further explore the forest. That is, the ability to obtain a set D_J, apply its corresponding function F_J, and produce the set of primitive Pythagorean triples T.

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