Department of Mathematics
California State University, Los Angeles

Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
SPRING 2020

Instructions:

- Do exactly two problems from part I AND two problems from part II. If you attempt more than two problems in either part I or part II, and do not clearly indicate which two are to count, only the first two problems will be counted towards your grade.
- No notes, books, calculators, or cell phones are allowed.

Part I: (Do exactly two problems)

1. (a) Without multiplying the factors of

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 5 \\ 0 & 0 & 1 \end{pmatrix}$$

justify the following:

i. The matrix A is non-singular. (4 points)

ii. The matrix A is symmetric. (4 points)

iii. The matrix A is positive definite. (4 points)

(b) What is this decomposition called? Are the matrices in this decomposition uniquely determined? (4 points)

(c) Count exact flops (all four arithmetic operations) required to obtain this decomposition of A. Show all steps with pertinent explanation. (6 points)

(d) Rewrite this decomposition in the form LDL^T where L is a lower triangular matrix and D is a diagonal matrix. (3 points)

2. (a) Given $A \in \mathbb{C}^{n\times n}$, and given $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$.

i. Describe the power method to find λ_1 and its corresponding eigenvector. (4 points)

ii. Show that the convergence is linear. (5 points)

iii. Identify one positive and one negative attribute of the power method, in comparison to the QR method. (3 points)

(b) i. Describe briefly how Rayleigh Quotient Iteration improves rate of convergence for power method. (3 points)

ii. Do two steps of Rayleigh Quotient Iteration on matrix $\begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$ (6 points) and compare the theoretical and observed rates of convergence (4 points).

3. Given the following matrix

$$A = \begin{pmatrix} 2 & s & s \\ s & 2 & 0 \\ s & 0 & 2 \end{pmatrix}$$

(a) Show that the spectral radius of the Jacobi iteration matrix for solving the system $A\mathbf{x} = \mathbf{b}$ is $\frac{|s|}{\sqrt{2}}$. (8 points)

(b) For what value(s) of s does the Jacobi iteration converge? What is the rate of convergence? (4 points)

(c) Let B be an arbitrary 4×4 upper triangular matrix with nonzero diagonal entries. Show that the Gauss-Seidel iteration converges when used to solve the system $B\mathbf{x} = \mathbf{b}$ for arbitrary vector $\mathbf{b}$. (7 points)

(d) If an iterative method solves a linear system with an accuracy 0.01 in 100 iterations, how many more iterations are needed to increase the accuracy to 0.00001? Explain your reasoning and simplify your answer. (6 points)

Part II: (Do exactly two problems)

1. Consider the PDE

$$\begin{aligned} U_t &= U_{xx}, \quad 0 \le x \le 1,\ t > 0 \\ U(x,0) &= x(1-x), \quad 0 \le x \le 1 \\ U_x(0,t) &= 0,\ U(1,t) = t, \quad t > 0. \end{aligned}$$

Suppose we approximate the above PDE by finite difference scheme

$$-r(1-\alpha)u_{i-1,j+1} + (1+2r\alpha)u_{i,j+1} - r(1-\alpha)u_{i+1,j+1} = r\alpha u_{i-1,j} + (1-2r(1-\alpha))u_{i,j} + r\alpha u_{i+1,j}$$

where $r = k/h^2$, $k = \Delta t$, $h = \Delta x$, $u_{i,j}$ approximates $U(ih, jk)$, and α is a parameter with $0 \le \alpha \le 1$.

(a) Find all value(s) of α for which the scheme is explicit. (3 points)

(b) Let $h = 1/3$, $k = 1/6$ (so $r = 3/2$) and $\alpha = 2/3$. Suppose the central difference approximation is used for the derivative boundary condition, find the matrices C and D, and vector $\mathbf{f}_j$ such that

$$C\mathbf{u}_{j+1} = D\mathbf{u}_j + \mathbf{f}_j.$$

Label clearly your matrices and vector $\mathbf{f}_j$, and indicate their dimensions. (10 points)

(c) Now let $\alpha = 1$. Perform von Neumann analysis to find the value(s) of r for which the scheme is stable. (6 points)

(d) Write (no proof) an explicit consistent scheme for approximating the following PDE. (6 points)

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{x^2+1}\frac{\partial u}{\partial x}\right)$$

2. Consider the ellptic PDE

$$U_{xx} + U_{yy} = f(x,y) \quad 0 < x < 3,\ 0 < y < 3.$$

(a) Write the standard 5-point finite difference stencil for approximating $U_{xx} + U_{yy}$ and calculate the order of accuracy of this approximation. (4 points)

(b) Suppose the boundary condition $U(x,y) = g(x,y)$ is prescribed for the above equation. Use the 5-point finite difference approximation from (a) to obtain a scheme for solving the boundary value problem on a rectangular grid with $\Delta x = \Delta y = 1$. Write the linear system in the matrix form $A\mathbf{u} = \mathbf{b}$. (6 points)

(c) Show that the system obtained from part (b) has a unique solution without solving the system. (4 points)

(d) Suppose the boundary condition $U_x(3,y) = 0,\ 0 < y < 3$ is prescribed along part of the boundary.

i. Use central difference to approximate the derivative boundary condition and write the linear system in the matrix form $B\mathbf{u} = \mathbf{c}$. Identify the dimension of this linear system. (7 points)

ii. Give another approach for approximating the derivative boundary condition. Briefly discuss the advantages and disadvantages of this approach in comparison to the central difference approximation. (4 points)

3. Consider the PDE

$$\begin{aligned} &U_t - aU_x = 0, \quad x \in \mathbb{R},\ t > 0 \\ &U(x,0) = e^{ax}\cos(2x), \quad x \in \mathbb{R}. \end{aligned}$$

Here a is a constant.

(a) Find the exact value of U at the point P that lies on the characteristic curve through $(3,0)$. (3 points)

(b) Use the method of characteristics to find the exact solution $U(x,t)$ to the problem. (4 points)

(c) i. For $a = -2$, write down the Upwind scheme for approximating the PDE. (5 points)

ii. Let $h = 1/2$, $k = 1/3$. Use the CFL condition to explain why or why not this scheme converges. (4 points)

(d) Consider the finite difference scheme

$$\frac{u_{i,j+1} - u_{i,j}}{k} = a\frac{u_{i+1,j} - u_{i-1,j}}{2h}$$

for approximating the PDE. Here $u_{i,j}$ approximates $U(x_i, t_j)$.

i. Show that the scheme is unstable when k is chosen proportional to h. (6 points)

ii. Can this scheme be made stable by adding correction term? Briefly explain your answer. (3 points)