Department of Mathematics
California State University Los Angeles

Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
FALL 2013

Instructions:

- Do exactly **two problems from Part A AND two problems from Part B**. If you attempt more than two problems in either Part A or Part B, and do not clearly indicate which two are to count, only the first two problems will be counted towards your grade.
- No calculators.
- Closed books and closed notes.

PART A: Do only **TWO** problems

1. Let

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}.$$

Note that A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -1$.

(a) [3 points] Find the eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ of A.

(b) [8 points] **Without** doing any iteration, verify that the Power Method will converge to the eigenvector of A associated with λ_1 when we take the initial vector $q_0 = (1, 0)^T$. Give an example of an initial vector for which it fails to converge. Briefly explain your reasoning.

(c) [3 points] Taking the initial iterate $q_0 = (1, 0)^T$, perform two iterations of the Power Method to obtain q_1, q_2.

(d) [4 points] Show that if $\langle \lambda, \mathbf{v} \rangle$ is an eigenpair for an $n \times n$ nonsingular matrix B, then $\langle 1/\lambda, \mathbf{v} \rangle$ is an eigenpair for B^{-1}.

(e) [4 points] Using part (d), explain how one can modify the Power Method so that it converges to the eigenvector associated with λ_2 of the matrix A given above.

(f) [3 points] Give one advantage and one disadvantage of the Power Method when used to find an approximation to the eigenvector.

2. (a) [4 points] What are the criteria for a Cholesky decomposition of a matrix A?

   (b) [4 points] Provide the algorithm for Cholesky decomposition.

   (c) [2 points] If the algorithm fails at any time, what can you conclude about matrix A?

   (d) [5 points] By going through a brief description of work involved, count how many flops the Cholesky algorithm requires.

   (e) [10 points] Solve the system $Ax = b$ using the Gaussian Elimination method with partial pivoting, where

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 4 & -1 & 6 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 10 \\ 11 \end{pmatrix}.$$

3. (a) [8 points] For

$$A = \begin{pmatrix} 1 & k & k \\ k & 1 & 0 \\ k & 0 & 1 \end{pmatrix},$$

   where $k > 0$, show that the spectral radius of Jacobi iteration matrix is $k\sqrt{2}$.

   (b) [5 points] For what values of k above does Jacobi iteration converge? What is the rate of convergence?

   (c) [8 points] For a given linear system $Ax = b$, and a splitting $A = M - N$, show that the iterative method $Mx^{(k+1)} = Nx^{(k)} + b$ converges linearly. What is the rate of convergence?

   (d) [4 points] If an iterative method solves a linear system $Ax = b$ with accuracy 10^{-2} in 230 iterations, then how many iterations will it need to increase accuracy to 10^{-3}? Explain briefly.

PART B: Do only **TWO** problems

1. Consider the following boundary value problem (BVP)

$$\begin{aligned} u_{xx} + u_{yy} &= 4, \quad 0 < x < 1, \quad 0 < y < 1 \\ u(x,0) = x^2, &\quad u(x,1) = (x-1)^2, \quad 0 \le x \le 1 \\ u(0,y) = y^2, &\quad u(1,y) = (y-1)^2, \quad 0 \le y \le 1 \end{aligned}$$

   (a) [3 points] Show that $u(x,y) = (x-y)^2$ is an exact solution to the above BVP.

   (b) [6 points] Show that the solution $u(x,y) = (x-y)^2$ is unique.

   (c) [10 points] Suppose we partition the domain $[0,1]\times[0,1]$ into a mesh with mesh size $h = \Delta x = \Delta y = 1/3$. Using central differences to approximate the derivatives, write a finite difference scheme to approximate the solution u at the resulting four interior mesh points. Simplify your answer in the matrix form $A\mathbf{u} = \mathbf{b}$.

   (d) [6 points] Suppose the boundary condition $u(x,1) = (x-1)^2$ is changed to $u_y(x,1) = 2x$. Write a second order finite difference formula to approximate the solution at the point $(2/3, 1)$ in the mesh of part (c).

2. Consider the initial boundary value problem (IBVP):

$$\begin{aligned} U_t &= \alpha U_{xx} - \beta U, &\quad 0 < x < 1,\ t > 0 \\ U(x,0) &= x(1-x), &\quad 0 \le x \le 1 \\ U(0,t) &= 0,\ U(1,t) = 0, &\quad t > 0 \end{aligned}$$

   where $\alpha > 0, \beta > 0$. An explicit approximation to the PDE with $r = k/h^2$ has the matrix form $\mathbf{u}_{j+1} = A\mathbf{u}_j$, where $\mathbf{u}_j = (u_{1,j}, u_{2,j}, \ldots, u_{N-1,j})^T$ and A is the tridiagonal matrix of order $N-1$:

$$A = \begin{pmatrix} 1-2r\alpha-k\beta & r\alpha & 0 & \cdots & 0 \\ r\alpha & 1-2r\alpha-k\beta & r\alpha & 0 & \vdots \\ 0 & r\alpha & 1-2r\alpha-k\beta & r\alpha & 0 \\ \vdots & \ddots & \ddots & \ddots & r\alpha \\ 0 & \cdots & 0 & r\alpha & 1-2r\alpha-k\beta \end{pmatrix}$$

   (a) [12 points] By determining the eigenvalues of the matrix A, obtain a restriction on r that guarantees that the scheme is stable.

   (b) [3 points] Write an expression for the given scheme solved for $u_{i,j+1}$ in terms of $u_{i-1,j}, u_{i,j}, u_{i+1,j}$.

   (c) [4 points] Give a consistent approximation to each of the initial and boundary conditions. (You need **not** show that your approximations are consistent.)

(d) [3 points] The truncation error for the given scheme is $T(h) = O(h^2)$. Use this fact to explain why the given scheme is consistent with the given PDE.

(e) [3 points] Assume that the given IBVP is well-posed. Explain why you can conclude from your work in the previous parts of this problem that the given scheme converges if r is restricted as found in part (a).

3. Consider the PDE

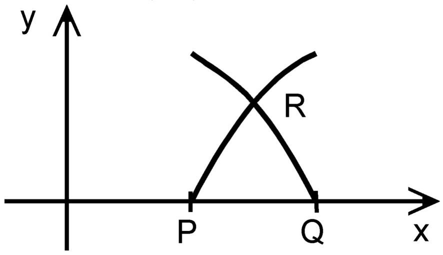
$$U_{xx} + xU_{xy} - 2x^2U_{yy} = 0$$

with initial data given on $y = 0$.

(a) [4 points] Determine all values of x for which the given PDE is hyperbolic.

(b) [5 points] Determine the two characteristic directions (slopes), dy/dx, for the given PDE at a general point (x, y).

(c) [6 points] Using the result of part (b), find the **exact values** of the coordinates of the point of intersection, R, of the characteristic curves through the points $P(1, 0)$



and $Q(2, 0)$.

(d) [2 points] Give the interval of dependence for $U(x, y)$ at the point R of part (c).

(e) [4 points] Derive a consistent finite difference approximation for the xU_{xy} term. (You need **not** show that your approximation is consistent.)

(f) [4 points] Suppose we approximate the given PDE by a consistent explicit difference scheme with $h = k = 1$. Referring to the CFL condition, explain why or why not this scheme converges at the point R of part (c).