Department of Mathematics
California State University, Los Angeles
Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
Fall 2011

Do exactly 2 problems from part I AND exactly two problems from part II.

Part I: (Do two problems)

1. Let A = $\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{bmatrix}$.

a. [2 pts] Give the value of $\|A\|_\infty$.

b. [2 pts] Is A *strictly* diagonally dominant (*yes* or *no*)?

c. [8 pts] Determine the LU decomposition of A.

d. [5 pts] Show that A is positive definite.

e. [8 pts] Let B_n denote the *n* x *n* matrix whose form is illustrated by $B_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{bmatrix}$

i.e., 1's on the diagonal, -1 in all below-diagonal positions, 1's in the last column, and all other entries 0. Show that if Gaussian elimination with partial pivoting is used, then B_n can be reduced to upper-triangular form without row exchanges, and the resulting matrix U has $u_{nn} = 2^{n-1}$

2. Let *A* be a 2x2 matrix with eigenvalues $\lambda_1 = 2, \lambda_2 = 1$ and corresponding eigenvectors $e_1 = \binom{1}{2}, e_2 = \binom{1}{1}$.

a. [5 pts] Find the matrix A.
b. [8 pts] Show that the Power Method will converge when applied to the matrix A above with initial vector $x^{(0)} = \binom{0}{2}$. What does it converge to? How do you modify the Power Method so that it will converge to the other eigenvalue?
c. [3 pts]Apply two iterations of the Power Method to the matrix *A*.

d. [3 pts] Give an example of an initial vector for which the Power Method will not converge. Briefly explain your reasoning.

e. [6 pts] For any nxn matrix *B* and any induced matrix norm, show that $\rho(B) \leq ||B||$.

3. Let $B = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{bmatrix}$

a. [9 pts] By computing the spectral radius of the iteration matrix, determine whether or not Jacobi iteration converges in solving $B\,\boldsymbol{x} = \mathbf{c}$, where **c** is an arbitrary 3-dimensional vector.

b. [9 pts] By computing the spectral radius of the iteration matrix, determine whether or not Gauss-Seidel iteration converges in solving $B\,\boldsymbol{x} = \mathbf{c}$, where **c** is an arbitrary 3-vdimensional ector.

c. [7 pts] Given the linear system $A\mathbf{x} = \mathbf{b}$ where *A* is an *n x n* matrix and **b** is an arbitrary *n*-dimensional vector, write *A* = *M* - *N*, where *M* is nonsingular, and consider the iterative method

$$\mathbf{x}^{(k+1)} = M^{-1}N\mathbf{x}^{(k)} + M^{-1}b \quad (k = 0, 1, 2, 3, \ldots)$$

Show that the error in the above approximation satisfies:

$$\left\|\mathbf{x}^{(k)} - \mathbf{x}\right\| \leq \|G\|^k \left\|\mathbf{x}^{(0)} - \mathbf{x}\right\|,$$

where $G = M^{-1}N$ and $\mathbf{x}^{(0)}$ is the initial approximation and **x** is the actual solution.

Part II: (Do two problems)

1. The unidirectional wave equation

$$u_t + 3u_x = 0$$
$$u(x,0) = f(x)$$

can be solved by using an upwind scheme

$$\frac{u_{i,j+1} - u_{i,j}}{k} + 3\frac{u_{i,j} - u_{i-1,j}}{h} = 0$$
$$u_{i,0} = f(ih)$$

where $u_{i,j} \approx u(ih, jk)$ and $h = \Delta x, k = \Delta t$ are the given grid sizes.

a. [5 pts] Find the CFL condition for the above scheme.
b. [10 pts] Perform the von Neumann analysis to show that the scheme is stable under the same condition found in part (a).
c. [10pts] Prove that under the same CFL condition, the scheme satisfies a local maximum-minimum principle, i.e.

$$\min(u_{i-1,j}, u_{i,j}) \le u_{i,j+1} \le \max(u_{i-1,j}, u_{i,j}).$$

2. Consider the PDE

$$u_t = u_{xx}, 0 \le x \le 1, t > 0$$
$$u(x,0) = x, 0 \le x \le 1$$
$$u(0,t) = u(1,t) = 0, t > 0$$

Suppose we approximate the above PDE by finite difference scheme

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2}$$

where $h = \Delta x, k = \Delta t$ are the given grid sizes.

a. [6 pts] Find the matrices B and C such that $B\boldsymbol{u}_{j+1} = C\boldsymbol{u}_j$ where $\boldsymbol{u}_j = [u_{1,j}, u_{2,j}, \dots, u_{N-1,j}]$.
b. [7 pts] Show that the scheme is consistent with the PDE. What is the order of accuracy of the scheme?
c. [8 pts] Does the scheme converge? Explain your answer.
d. [4 pts] Suppose we approximate the above PDE by a θ-method:

$$-r\theta u_{i-1,j+1} + (1 + 2r\theta)u_{i,j+1} - r\theta u_{i+1,j+1}$$
$$= r(1-\theta)u_{i-1,j} + \big(1 - 2r(1-\theta)\big)u_{i,j} + r(1-\theta)u_{i+1,j}.$$

For what value(s) of θ is this scheme implicit?

3. Consider the following boundary value problem:

$$2u_{xx} + u_{yy} = 6 \quad 0 < x < 1,\ 0 < y < 1,$$

$$u(x, 0) = x^2,\ u(x, 1) = (x - 1)^2$$

$$u(0, y) = y^2,\ u(1, y) = (1 - y)^2$$

a. [5 pts] Show that $u(x, y) = (x - y)^2$ is a solution to the boundary value problem.

b. [10 pts] Using central differencing for the derivatives write down a finite difference scheme for the problem and simplify it. Show that your scheme is consistent.

c. [10 pts] Using $\Delta x = \Delta y = 1/3$ obtain a system of linear equation for solving this problem. Express this in the form A**u** = **b** where A is a 4 x 4 matrix.