Department of Mathematics
California State University Los Angeles

Master's Degree Comprehensive Examination in

NUMERICAL ANALYSIS
SPRING 2022

Instructions:

- Do exactly **two problems from Part A AND two problems from Part B**. If you attempt more than two problems in either Part A or Part B, and do not clearly indicate which two are to count, only the first two problems will be counted towards your grade.
- No notes, books, calculators, internet or cell phones are allowed.

PART A: Do only **TWO** problems

1. (a) [6 pts] Factor

$$A = \begin{pmatrix} 3 & 1 \\ 4 & 1 \end{pmatrix}$$

into $A = QR$, where Q is an orthogonal matrix and R is an upper-triangular matrix.

(b) [7 pts] Let M be an $n \times n$ symmetric matrix. Briefly explain the QR algorithm for finding the eigenvalues of M. Can this algorithm find the eigenvectors of M as well? Explain your answer.

(c) [6 pts] Let M_k be the k-th iterate of the QR algorithm. Show that M_k is similar to M.

(d) [6 pts] Let $\mathbf{x}^{(0)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Find $\mathbf{x}^{(1)}$ and $\mathbf{x}^{(2)}$ using the Power method for approximating the dominant eigenvalue of the matrix A in part (a).

2. (a) [8 pts] By computing the eigenvalues of the iteration matrix, determine whether or not the Gauss-Seidel method converges when applied to the system $B\mathbf{x} = \mathbf{c}$, where

$$B = \begin{pmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \text{ and } \mathbf{c} \text{ is an arbitrary vector.}$$

(b) [7 pts] Find k such that $\|\mathbf{x}^{(k)} - \mathbf{x}\| \leq 10^{-3}\|\mathbf{x}^{(0)} - \mathbf{x}\|$, where $\mathbf{x}^{(k)}$ is the k-th iterate of the Gauss-Seidel iteration in part (a).

(c) [10 pts] For a given linear system $A\mathbf{x} = \mathbf{b}$, and a splitting $A = M - N$, show that the iterative method $M\mathbf{x}^{(k+1)} = N\mathbf{x}^{(k)} + \mathbf{b}$ converges linearly if $\rho(G) < 1$, where $G = M^{-1}N$ and $\rho(G)$ is the spectral radius of G. You may assume that the eigenvectors of G are linearly independent.

3. Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 2 & 2 & -1 \\ 4 & -1 & 6 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 3 \\ 10 \\ 11 \end{pmatrix}.$$

(a) [8 pts] Solve $A\mathbf{x} = \mathbf{b}$ using Gaussian elimination with partial pivoting, if possible. If not possible, explain your reasoning.

(b) [5 pts] Write $PA = LU$, where P is the appropriate permutation matrix. Is this factorization unique? Why or why not?

(c) [8 pts] Write the Cholesky factors of A, if possible. If not possible, explain your reasoning. From your result conclude whether or not A is positive definite.

(d) [4 pts] Suppose an $n \times n$ matrix B has been factored into $B = R^T R$, where R is an upper triangular matrix. Explain how you can use this factorization to efficiently solve the system $B\mathbf{x} = \mathbf{c}$.

PART B: Do only **TWO** problems

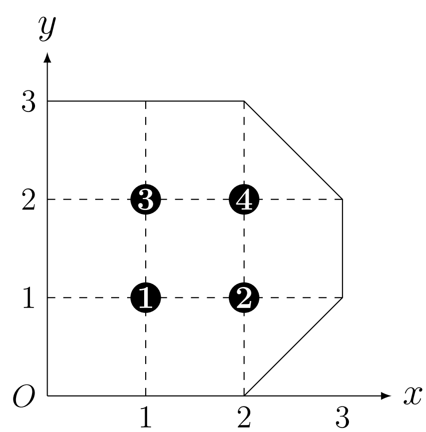
1. (a) [6 pts] Find and sketch the regions of hyperbolicity, ellipticity and parabolicity for the PDE
$$u_{xx} + 2xu_{xy} + (x + y)u_{yy} = u. \tag{1}$$
(b) [5 pts] Derive a consistent finite difference approximation for the term $2xu_{xy}$ that is first order accurate. You need not prove that your approximation is consistent.

(c) [5 pts] Find the partial derivative which is approximated by the finite difference
$$\frac{u_{i+2,j} + 2u_{i+1,j} - 2u_{i-1,j} - u_{i-2,j}}{8\Delta x}. \tag{2}$$
What is the order of accuracy to this approximation?

(d) Given the hyperbolic PDE
$$u_{xx} - 4x^2 u_{yy} = 0$$
$$u(x, 0) = x^2, \ -\infty < x < \infty$$
$$u_y(x, 0) = 0, \ -\infty < x < \infty$$
i. [3 pts] Find the slope of the characteristic curves of this PDE.

ii. [6 pts] Suppose the characteristic curves that pass through the points $P(-0.2, 0)$ and $Q(-0.4, 0)$ intersect at a point $R(x_R, y_R)$, where $y_R > 0$. Find the exact values of x_R and y_R.

2. Consider the PDE $u_{xx} + u_{yy} = 0$ with boundary values $u(x, y) = x^2 + y^2$ defined on the region R shown below:



(a) [5 pts] Find the maximum value of u attained within the region R. At what point(s) is this maximum value attained?

(b) [8 pts] Suppose that we approximate this PDE by the usual 5-point scheme on a square mesh with $h = 1$. Write down the resulting system of linear equations in matrix form $A\mathbf{u} = \mathbf{b}$. Use the node labeling as shown in the above figure.

(c) [5 pts] Without solving the system, explain whether the system $A\mathbf{u} = \mathbf{b}$ found in part (b) has a unique solution. Briefly explain your reasoning.

(d) [7 pts] Now suppose that the boundary condition along $x = 0$ is replaced by $u_x(0, y) = 2$ for $0 < y < 3$. Write a second order accurate finite difference equation to approximate the solution at the point $(0, 1)$ in terms of $u(0, 2)$ and $u(1, 1)$.

3. Consider the PDE

$$\begin{aligned} u_t &= u_{xx}, \;\; 0 \le x \le 1 \\ u(x,0) &= x, \;\; 0 \le x \le 1 \\ u(0,t) &= u(1,t) = 1, \;\; t > 0. \end{aligned}$$

Suppose we approximate the above PDE by the finite difference scheme

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2},$$

where $h = \Delta x$ and $k = \Delta t$ are the given grid sizes.

(a) [5 pts] Find the matrices A, B and vector $\mathbf{f}_j$ such that $A\mathbf{u}_{j+1} = B\mathbf{u}_j + \mathbf{f}_j$, where $\mathbf{u}_j = (u_{1,j}, u_{2,j}, \ldots, u_{N-1,j})^T$.

(b) [8 pts] Show that the scheme is consistent with the PDE. What is the order of accuracy of the scheme?

(c) [8 pts] Does the scheme converge? Explain your answer.

(d) [4 pts] Suppose we approximate the above PDE by a θ-method:

$$-r(1-\theta)u_{i-1,j+1} + (1+2r(1-\theta))u_{i,j+1} - r(1-\theta)u_{i+1,j+1} = r\theta u_{i-1,j} + (1-2r\theta)u_{i,j} + r\theta u_{i+1,j}.$$

For what value(s) of θ is this scheme implicit?