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MATH 540A
Due: 11/28/07

5.4-A 6.1-A,B

5.4

A) [Classify all group of size $5^2 \cdot 7$.]

Let's first show that if $|G| = 5^2 \cdot 7$, then G is abelian. Let $P \in Syl_5(G)$ and $Q \in Syl_7(G)$, so that $|P| = 5^2$ and $|Q| = 7$. We must have that $n_5 \equiv 1 \pmod 5$ and $n_5 \mid 7$, so we must have $n_5 = 1$. Thus, $P \trianglelefteq G$. Also, $n_7 \equiv 1 \pmod 7$ and $n_7 \mid 5^2$, so $n_7 = 1$. Thus, $Q \trianglelefteq G$. Since $P \cap Q \leq P$ and $P \cap Q \leq Q$, $|P \cap Q|$ must divide $|P| = 5^2$ and $|Q| = 7$. So, $|P \cap Q| = 1$. Thus, $P \cap Q = \{1\}$.

Now we show that $G' = \{1\}$ by showing that $G' \leq P \cap Q$. Since $|G/P| = \frac{|G|}{|P|} = \frac{5^2 \cdot 7}{5^2} = 7$, G/P is cyclic and therefore abelian. By Prop 7 (p. 169), since $P \trianglelefteq G$ and G/P is abelian, $G' \leq P$. Also, since $|G/Q| = \frac{|G|}{|Q|} = \frac{5^2 \cdot 7}{7} = 5^2$, G/Q is abelian (by thm proved in class). So, $G' \leq Q$. Thus, $G' \leq P \cap Q$, so $G' = \{1\}$. Thus, G is abelian.

Since G is abelian, the FTOFGAG says that $\boxed{G \cong \mathbb{Z}_{5^2 \cdot 7} \text{ or } G \cong \mathbb{Z}_{5 \cdot 7} \times \mathbb{Z}_5}$.

DB2

6.1 A) [Let G be a p-group. Prove that G is solvable. (Hint: Induct on $|G|$, and think about $Z(G)$.)]

Pf: Proceed by induction on $|G| = p^\alpha$, $\alpha \geq 0$.

$\alpha = 0$: Then $|G| = p^0 = 1$, so G is solvable (since $\{1\} \trianglelefteq G$ and $G/\{1\} \cong \{1\}$, which is abelian).

Suppose that groups of order p^α are solvable. Let $|G| = p^{\alpha+1}$. Then consider $Z(G)$. Since G is a p-group, $|Z(G)| > 1$. By Lagrange, $|Z(G)| = p^\beta$ for $1 \leq \beta < \alpha$. Thus,

$$\left| G/Z(G) \right| = \frac{|G|}{|Z(G)|} = \frac{p^{\alpha+1}}{p^\beta} = p^{\alpha-\beta+1}$$

Since $\alpha - \beta + 1 \leq \alpha$, $G/Z(G)$ is solvable. Also, $Z(G) \trianglelefteq G$, and since $Z(G)$ is abelian, it is solvable. Thus, by Prop 10, G is solvable.

By induction, any p-group is solvable. ▨

DB3

(6.1 cont) B) [Show that a group of size $3^3 \cdot 11^4$ is solvable.]

Pf: Let $|G| = 3^3 \cdot 11^4$. Let $Q \in Syl_{11}(G)$. Then since Q is an 11-group, it is solvable by part (A). Also, $n_{11} \equiv 1 \pmod{11}$ and $n_{11} \mid 3^3 = 27$. Thus $n_{11} = 1$. So, $Q \trianglelefteq G$. Now consider G/Q. Since $|G/Q| = \frac{|G|}{|Q|} = \frac{3^3 \cdot 11^4}{11^4} = 3^3$. Since G/Q is a p-group, it is solvable by part (A). Thus, by Proposition 10, G is solvable.

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