$\boxed{\begin{array}{c} 9/9/19 \\ \text{Monday} \end{array}}$

$\boxed{\text{HW 2}}$

9(c) $A_n = \left(2+\frac{1}{n}, n\right) \subseteq \mathbb{R}$

Calculate $\bigcup_{n=3}^{\infty} A_n$ and $\bigcap_{n=3}^{\infty} A_n$

$$\bigcup_{n=3}^{\infty} A_n = (2, \infty)$$

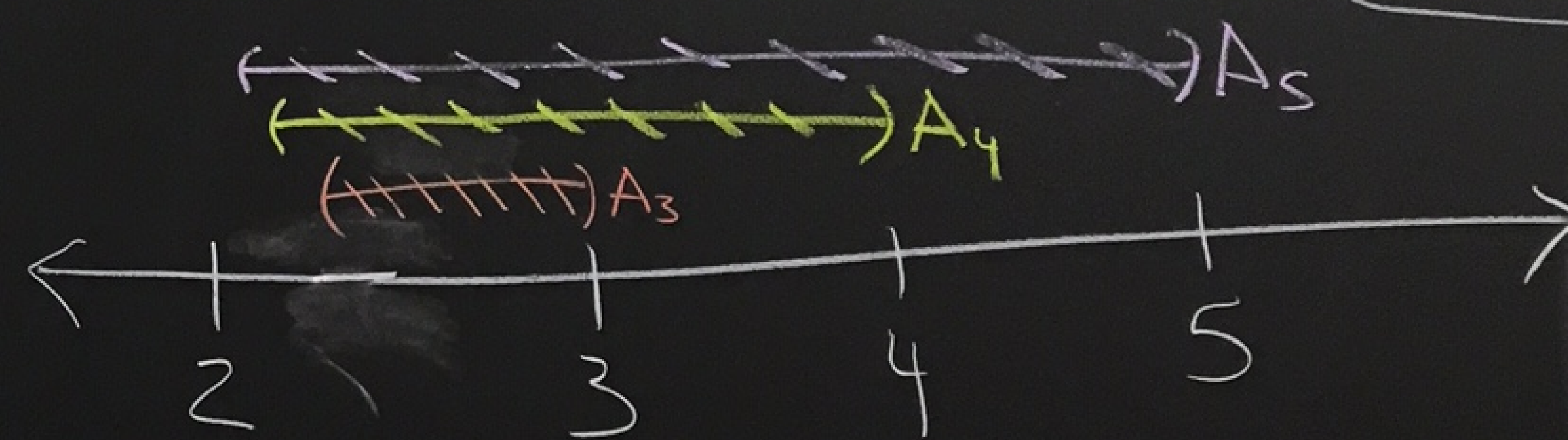
$$\bigcap_{n=3}^{\infty} A_n = A_3 = \left(2+\frac{1}{3}, 3\right) = \left(\frac{7}{3}, 3\right)$$

$$A_3 = \left(2+\frac{1}{3}, 3\right)$$

$$A_4 = \left(2+\frac{1}{4}, 4\right)$$

$$A_5 = \left(2+\frac{1}{5}, 5\right)$$

$$\vdots \qquad \vdots$$



From last time

Prop: Let $\mathcal{A} = \{A_\alpha \mid \alpha \in \Delta\}$ be an indexed family of sets. Pick $\alpha_0 \in \Delta$. Then:

① $A_{\alpha_0} \subseteq \bigcup_{\alpha \in \Delta} A_\alpha$

② $\bigcap_{\alpha \in \Delta} A_\alpha \subseteq A_{\alpha_0}$

pf:

① Let $x \in A_{\alpha_0}$.

Recall that

$$\bigcup_{\alpha \in \Delta} A_\alpha = \left\{ y \;\middle|\; \begin{array}{l} \text{there exists } \alpha' \in \Delta \\ \text{where } y \in A_{\alpha'} \end{array} \right\}$$

Since $x \in A_{\alpha_0}$ there does indeed exist $\alpha' \in \Delta$ with $x \in A_{\alpha'}$, ie $\alpha' = \alpha_0$.

So $x \in \bigcup_{\alpha \in \Delta} A_\alpha$.

Thus, $A_{\alpha_0} \subseteq \bigcup_{\alpha \in \Delta} A_\alpha$.

② You try.

$\blacksquare$

(HW 3 TOPIC) — Equivalence relations & Well-defined operations & Modulo n

<u>Def:</u> A <u>relation</u> $\sim$ on a set S is a subset of $S \times S$. If (x,y) is an element of $\sim$ then we write $x \sim y$ and say that <u>x is related to y</u>.
If (x,y) is not an element of $\sim$ then we write $x \not\sim y$ and say that <u>x is not related to y</u>.

Ex: $S = \{\square, \$, ¢\}$

$\sim = \{(\square, ¢), (\square, \square), (\square, \$), (¢, \square), (¢, \$)\}$

$\square \sim ¢$ since $(\square, ¢) \in \sim$.

$¢ \sim \square$ since $(¢, \square) \in \sim$.

$\square \sim \square$ since $(\square, \square) \in \sim$.

$\$ \nsim \$$ since $(\$, \$) \notin \sim$.

More common way to define a relation is like the following example.

Ex: $S = \mathbb{Z}$

Define $x \sim y$ to mean $x < y$.

Then $\sim$ is a relation on $\mathbb{Z}$.

We have

$2 \sim 3$ since $2 < 3$.

$2 \not\sim -1$ since $2 \not< -1$.

How can we think of $\sim$ as a subset of $\mathbb{Z} \times \mathbb{Z}$?

$$\sim \; = \{(x,y) \mid x < y\}$$
$$= \{(2,3), (-100,0), (7,777), \ldots\}$$

infinitely many more elements

can think of an equivalence relation as an "equals" sign.

Def: Let $\sim$ be a relation on a set S.

We say that $\sim$ is an <u>equivalence relation</u> on S if

1. (reflexive) $x \sim x$ for all $x \in S$.
2. (symmetric) If $x, y \in S$ and $x \sim y$, then $y \sim x$.
3. (transitive) If $x, y, z \in S$ and $x \sim y$ and $y \sim z$, then $x \sim z$.

$\underline{\text{Ex}}$: Let $S = \{-15, 2, 0\}$

and $\sim \, = \{(0,0), (0,2), (2,0), (-15,2)\}$

(reflexive?) $0 \sim 0$ but $2 \not\sim 2$ and $(-15) \not\sim (-15)$.

So, $\sim$ is not reflexive.

(symmetric?) NO! $(-15) \sim 2$ but $2 \not\sim (-15)$.

(transitive?)

$0 \sim 0$ and $0 \sim 2 \Rightarrow 0 \sim 2$ ✓

$0 \sim 2$ and $2 \sim 0 \Rightarrow 0 \sim 0$ ✓

$2 \sim 0$ and $0 \sim 0 \Rightarrow 2 \sim 0$ ✓

$0 \sim 0$ and $0 \sim 0 \Rightarrow 0 \sim 0$ ✓

$\boxed{-15 \sim 2 \text{ and } 2 \sim 0 \text{ but } -15 \not\sim 0}$

So, $\sim$ is not transitive.

$\boxed{\sim \text{ is NOT an equivalence relation}}$

Ex: $S = \{1, 2, 3\}$

$\sim = \{(1,1), (2,2), (3,3), (1,3), (3,1)\}$

Is $\sim$ an equivalence relation? YES

reflexive

$1 \sim 1$

$2 \sim 2$

$3 \sim 3$

Yes

symmetric

Yes

If $x \sim y$, then $y \sim x$.

Ex: $1 \sim 3$ and $3 \sim 1$.

transitive

- $1 \sim 1$ and $1 \sim 1$. And $1 \sim 1$.
- $1 \sim 1$ and $1 \sim 3$. And $1 \sim 3$.
- $2 \sim 2$ and $2 \sim 2$. And $2 \sim 2$.
- $3 \sim 3$ and $3 \sim 3$. And $3 \sim 3$.
- $3 \sim 3$ and $3 \sim 1$. And $3 \sim 1$.
- $1 \sim 3$ and $3 \sim 3$. And $1 \sim 3$.
- $1 \sim 3$ and $3 \sim 1$. And $1 \sim 1$.
- $3 \sim 1$ and $1 \sim 1$. And $3 \sim 1$.
- $3 \sim 1$ and $1 \sim 3$. And $3 \sim 3$.

Yes

Def: Let $\sim$ be an equivalence relation on a set S. Let $x \in S$. The *equivalence class* of x is

$$\overline{x} = \{ y \in S \mid x \sim y \}$$

$\overline{x}$ consists of all elements y that are related to x.

It doesn't matter if you define $\overline{x}$ as $\{ y \in S \mid x \sim y \}$ or $\{ y \in S \mid y \sim x \}$ since $\sim$ is symmetric.

Another notation for $\overline{x}$ is $[x]$.

Ex: $S = \{1, 2, 3\}$

$$\sim \; = \{(1,1), (2,2), (3,3), (1,3), (3,1)\}$$

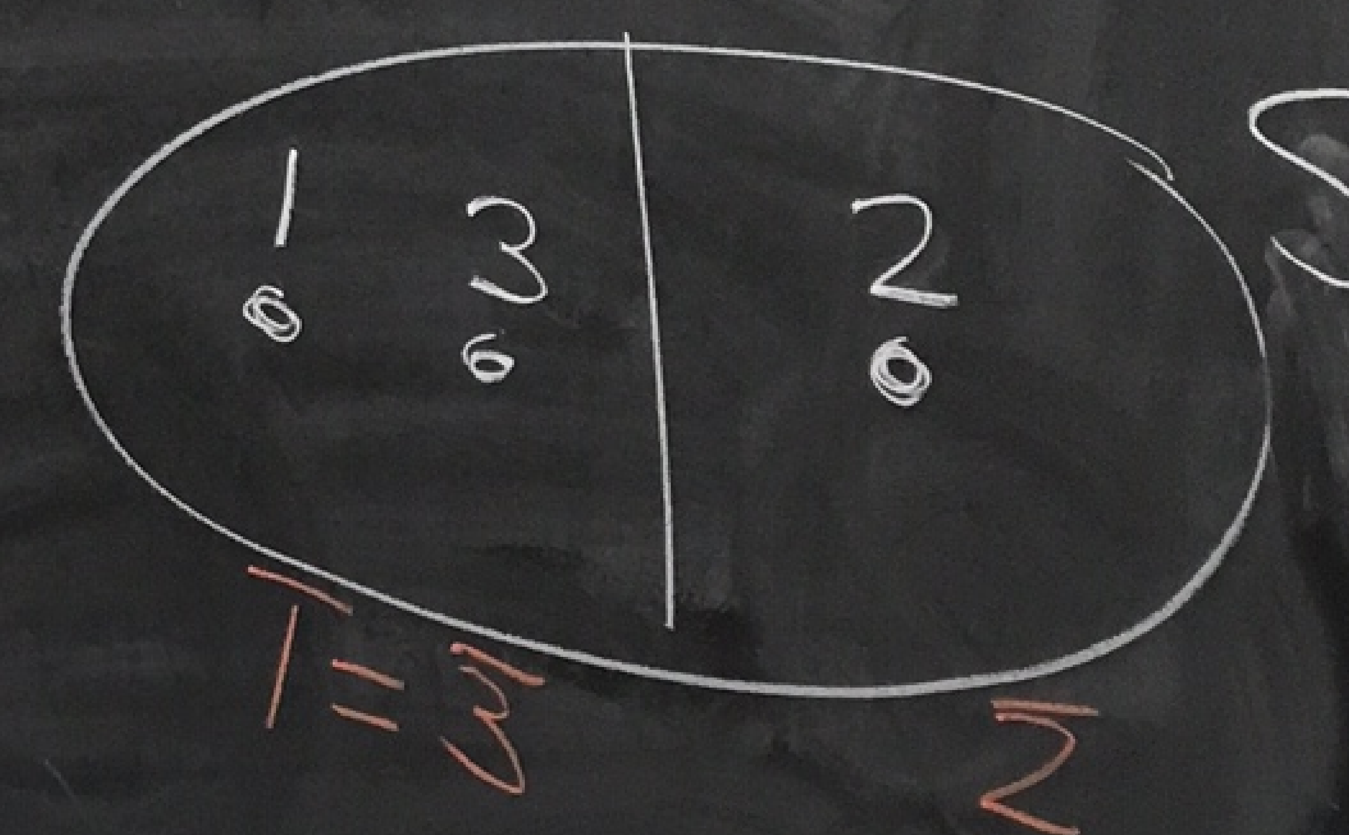
$\sim$ is an equivalence relation.

$$\overline{1} = \{y \in S \mid 1 \sim y\} = \{1, 3\}$$

$$\overline{2} = \{y \in S \mid 2 \sim y\} = \{2\}$$

$$\overline{3} = \{y \in S \mid 3 \sim y\} = \{1, 3\}$$

There are 2 equivalence classes $\overline{1} = \overline{3}$ and $\overline{2}$



Def: Let S be a set and $\sim$ be an equivalence relation on S. We denote the set of equivalence classes as $S/\sim$ read: "S mod $\sim$"

Ex: In the previous example

$$S/\sim \;=\; \{\bar{1}, \bar{2}\}$$

$\bar{1} = \bar{3}$

Idea:

since $1 \sim 3$ we make 1 and 3 equal in the set of equivalence classes