Weds 9/4

Recall: Let A be a set.

The power set of A is

$$\mathcal{P}(A) = \{ B \mid B \subseteq A \}$$

$$\mathcal{P}(\{a,b\}) = \{ \emptyset, \{a\}, \{b\}, \{a,b\} \}$$

Theorem: Let X and Y be sets. Then $X = Y$ if and only if $\mathcal{P}(X) = \mathcal{P}(Y)$.

proof:

$(\Rightarrow)$ If $X = Y$, then $\mathcal{P}(X) = \mathcal{P}(Y)$.

$(\Leftarrow)$ Suppose $\mathcal{P}(X) = \mathcal{P}(Y)$.

We want to show that $X = Y$.

To show $X = Y$ we will show $X \subseteq Y$ and $Y \subseteq X$.

$\boxed{X \subseteq Y}$:

We know that $X \subseteq X$.

So $X \in \mathcal{P}(X)$.

But $\mathcal{P}(X) = \mathcal{P}(Y)$ so $X \in \mathcal{P}(Y)$.

So, $X \subseteq Y$.

$\boxed{Y \subseteq X}$: The same proof as above with X & Y interchanged shows that $Y \subseteq X$.

Since $X \subseteq Y$ and $Y \subseteq X$ we have $X = Y$.

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A different approach to show $X \subseteq Y$ in the previous proof:

Let $a \in X$.

Then $\{a\} \subseteq X$ so $\{a\} \in \mathcal{P}(X)$.

Since $\mathcal{P}(X) = \mathcal{P}(Y)$ we have $\{a\} \in \mathcal{P}(Y)$.

Thus, $\{a\} \subseteq Y$.

Then $a \in Y$.

We have shown $X \subseteq Y$.

Families of sets

Def: When every element of a set $\mathcal{A}$ is itself a set then we call $\mathcal{A}$ a <u>family</u> or <u>collection</u> of sets.

Ex: $\mathcal{P}(X)$ is a family of sets

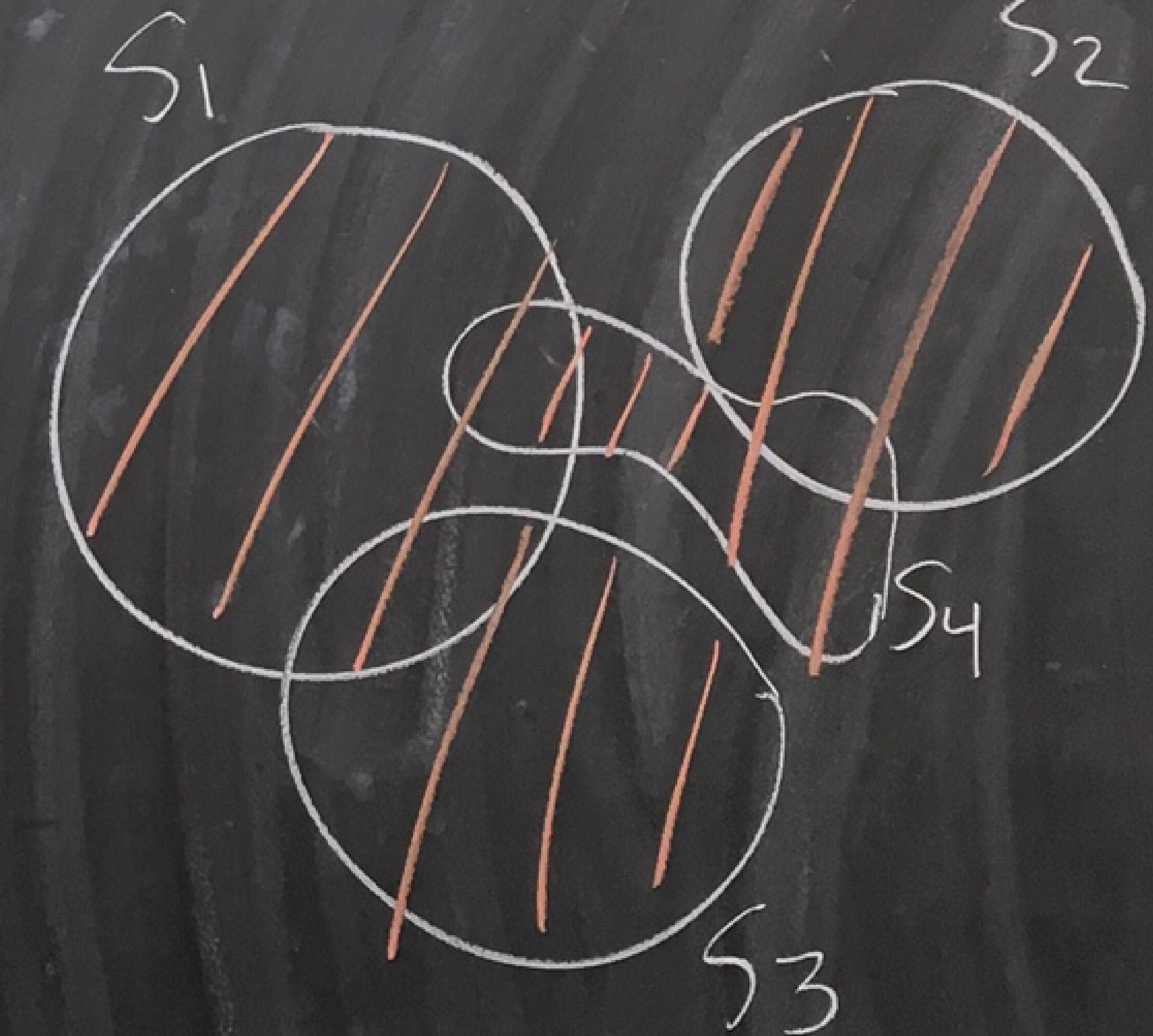
Def: Let $\mathcal{A}$ be a non-empty family of sets.

We define the union over $\mathcal{A}$ to be

$$\bigcup_{S\in\mathcal{A}} S = \{x \mid x\in S \text{ for some } S\in\mathcal{A}\}$$

$$= \{x \mid \text{there exists } S\in\mathcal{A} \text{ with } x\in S\}$$

$\mathcal{A}=\{S_1,S_2,S_3,S_4\}$

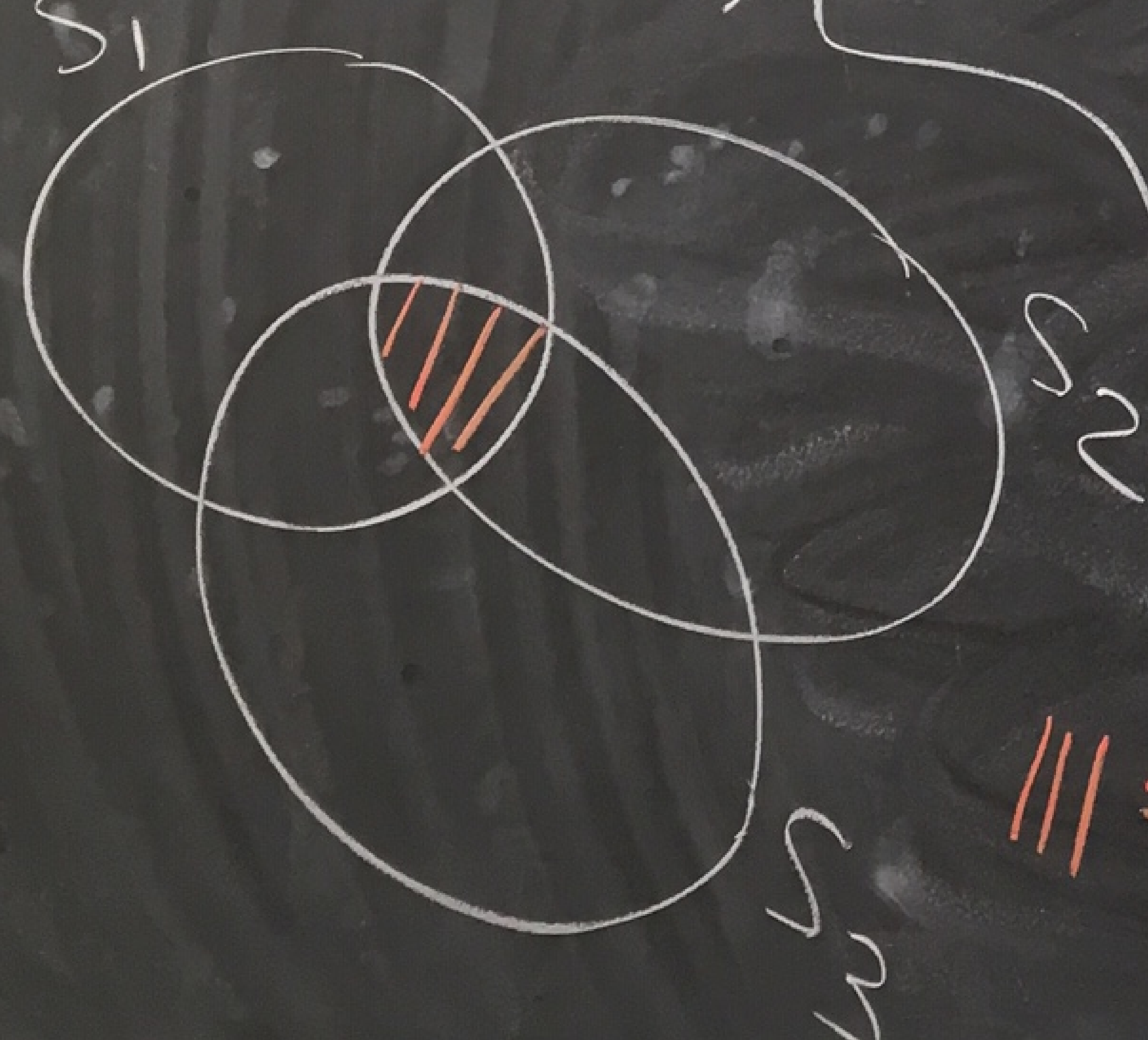


$/\!/\!/ = \bigcup_{S\in\mathcal{A}} S$

We define the <u>intersection over A</u> to be

$$\bigcap_{S\in A} S = \{x \mid x\in S \text{ for all } S\in A\}$$

$A=\{S_1,S_2,S_3\}$



$/// = \bigcap_{S\in A} S$

Ex:

$$A=\{\{1,\sqrt{2},3,e^4\}, \{5,3,e^4,\sqrt{6},78\}, \{i+1,1,10\}\}$$

e

$A\}$

$$\bigcup_{S\in A} S = \{1, \sqrt{2}, 3, e^4, 5, \sqrt{6}, 78, i+1, 10\}$$

$$\bigcap_{S\in A} S = \phi$$

$, 78\},$

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Ex: $\mathcal{B} = \{\{n \in \mathbb{Z} \mid |n| \leq k\} \mid k \in \mathbb{N}\}$

$= \{S_k \mid k \in \mathbb{N}\} = \{S_1, S_2, S_3, \ldots\}$

where $S_k = \{n \in \mathbb{Z} \mid |n| \leq k\}$

$S_1 = \{-1, 0, 1\}$

$S_2 = \{-2, -1, 0, 1, 2\}$

$S_3 = \{-3, -2, -1, 0, 1, 2, 3\}$

$$\bigcup_{S \in \mathcal{B}} S = \mathbb{Z} = \{\ldots, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, \ldots\}$$

$$\bigcap_{S \in \mathcal{B}} S = S_1 = \{-1, 0, 1\}$$

<u>Def:</u> Let Δ be a non-empty set. Suppose that for each $\alpha \in \Delta$ there is a corresponding set A_α. The family

$\mathcal{A} = \{ A_\alpha \mid \alpha \in \Delta \}$ is called an <u>indexed family of sets</u>. The set Δ is called the <u>index set</u>. If $\alpha \in \Delta$ then α is called the index of A_α.

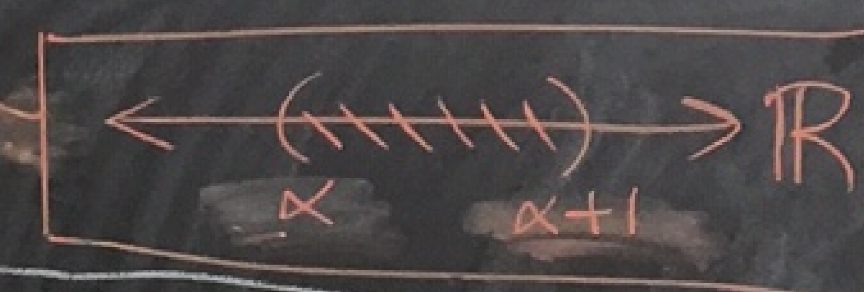
Given such a family of sets define

$$\bigcup_{\alpha \in \Delta} A_\alpha = \bigcup_{S \in \mathcal{A}} S = \left\{ x \;\middle|\; \begin{array}{l} \text{there exists } \alpha \in \Delta \\ \text{with } x \in A_\alpha \end{array} \right\}$$

$$\bigcap_{\alpha \in \Delta} A_\alpha = \bigcap_{S \in \mathcal{A}} S = \left\{ x \;\middle|\; \begin{array}{l} x \in A_\alpha \text{ for} \\ \text{all } \alpha \in \Delta \end{array} \right\}$$

define these in terms of previous def.

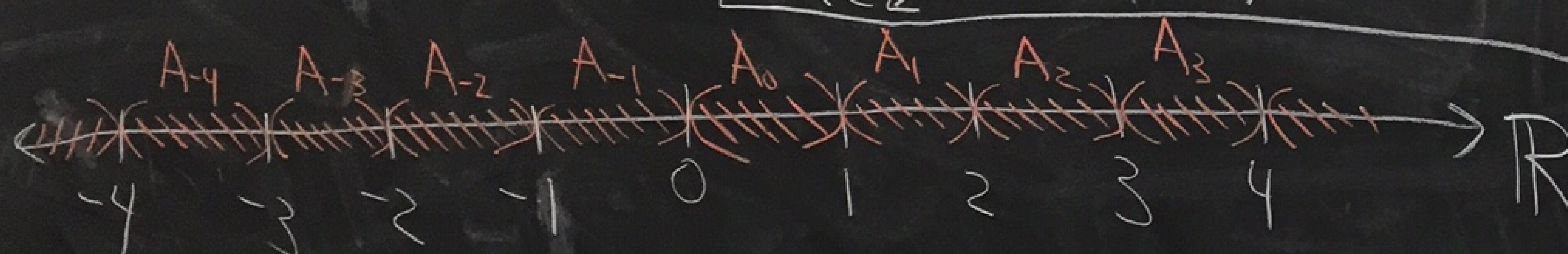
Ex: Let's make sense of $\bigcup\limits_{\alpha\in\mathbb{Z}} (\alpha,\alpha+1)$ in terms of the previous definition.



$\Delta = \mathbb{Z}$

Given $\alpha \in \Delta$,
set $A_\alpha = (\alpha, \alpha+1)$

$$\boxed{\begin{aligned} \bigcup_{\alpha\in\mathbb{Z}} (\alpha,\alpha+1) &= \mathbb{R} - \mathbb{Z} \\ \bigcap_{\alpha\in\mathbb{Z}} (\alpha,\alpha+1) &= \emptyset \end{aligned}}$$



Some people write these sets like this:

$$\boxed{\bigcup_{\alpha=-\infty}^{\infty} (\alpha,\alpha+1)}$$

$$\boxed{\bigcap_{\alpha=-\infty}^{\infty} (\alpha,\alpha+1)}$$

Its understood that α ranges over $\mathbb{Z}$

Its understood that α ranges over $\mathbb{Z}$

<u>Prop:</u> Let $\mathcal{A} = \{A_\alpha \mid \alpha \in \Delta\}$ be an indexed family of sets. Pick some $\alpha_0 \in \Delta$. Then:

① $A_{\alpha_0} \subseteq \bigcup_{\alpha \in \Delta} A_\alpha$

② $\bigcap_{\alpha \in \Delta} A_\alpha \subseteq A_{\alpha_0}$