Weds 9/25

$\mathbb{N} = \{1, 2, 3, 4, \dots\}$

HW 3

8 $S = \mathbb{N} \times \mathbb{N} = \{(a,b) \mid a \in \mathbb{N}, b \in \mathbb{N}\}$

$= \{(1,1), (2,5), (103,301), \dots\}$

Define $\sim$ on S by

$(a,b) \sim (c,d)$ means $a+d = b+c$

(a) Is $(3,6) \sim (7,10)$?

$3+10 = 6+7$.

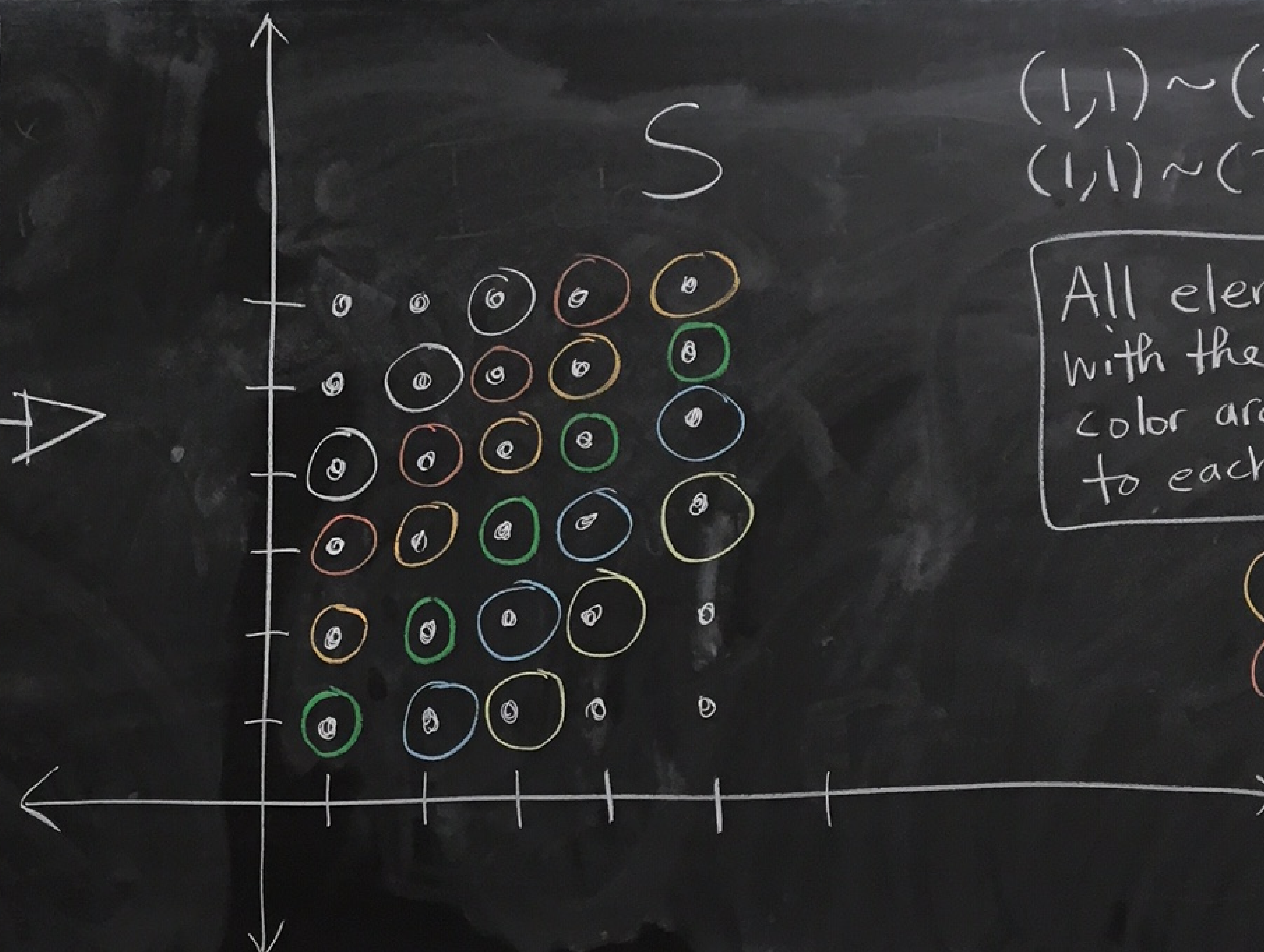
Yes $(3,6) \sim (7,10)$

(b) Is $(1,1) \sim (3,5)$?

No, $1+5 \neq 1+3$.

PICTURE TIME $\longrightarrow$

$(1,3) \sim (2,4)$



$(1,1) \sim (2,2)$

$(1,1) \sim (3,3)$

All elements with the same color are related to each other

$(1,2) \sim (2,3)$

$(1,2) \not\sim (2,1)$

$(1,2) \sim (1,2)$

$\boxed{\begin{array}{l}(a,b)\sim(c,d)\\ a+d=b+c\end{array}}$

(c) Prove that $\sim$ is an equivalence relation on S.

(reflexive) Let $x \in S$.
Then $x=(a,b)$ where $a,b\in\mathbb{N}$
Since $a+b=b+a$ we know
$(a,b)\sim(a,b)$.
So, $x\sim x$.

(symmetric) Let $(a,b),(c,d)\in S$
where $a,b,c,d\in\mathbb{N}$.
And suppose $(a,b)\sim(c,d)$.
Then $a+d=b+c$.
So, $c+b=d+a$.
Thus, $(c,d)\sim(a,b)$.

(transitive) Let $(a,b),(c,d),(e,f) \in S$.

Assume that $(a,b)\sim(c,d)$ and $(c,d)\sim(e,f)$.

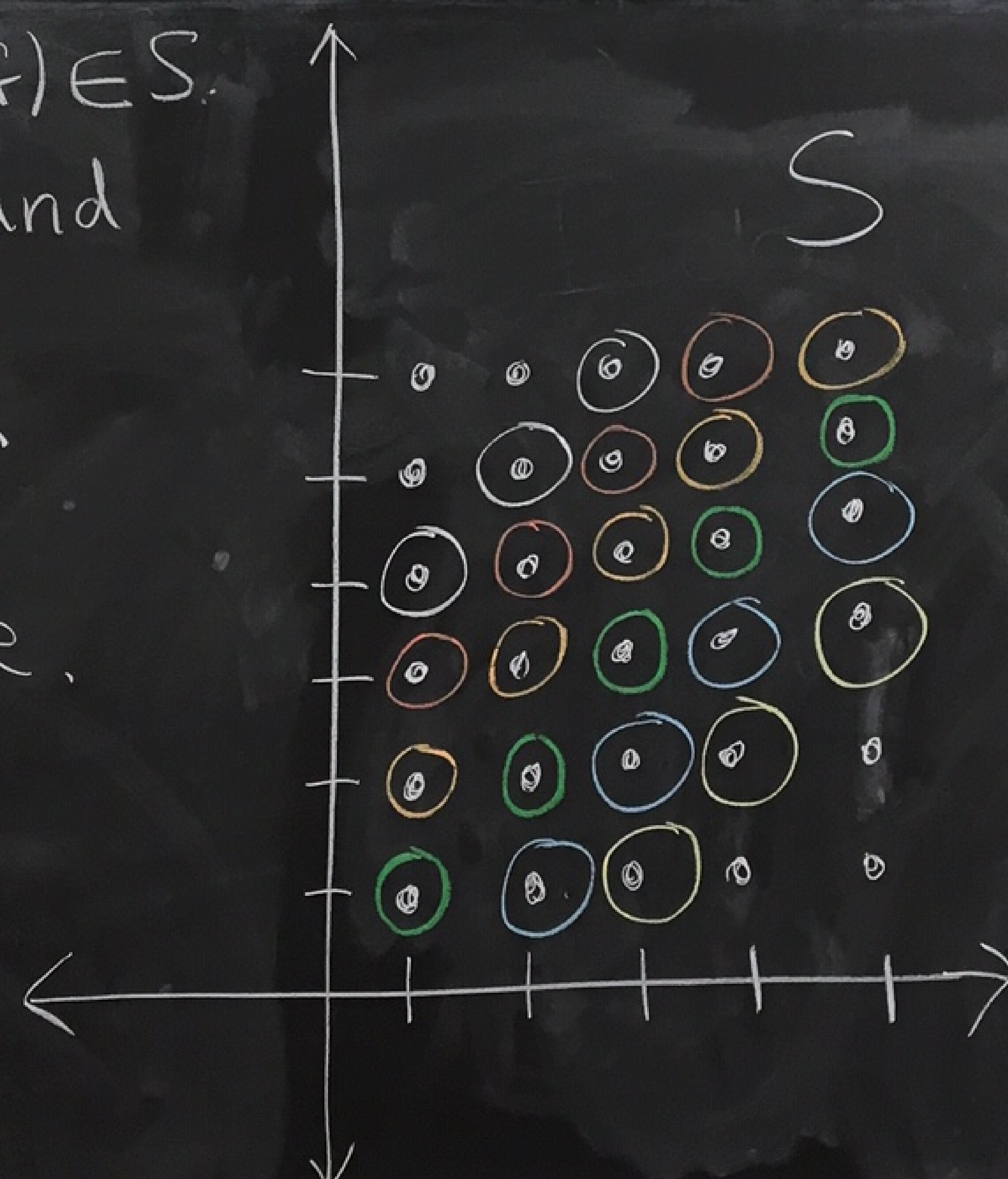
Then, $a+d=b+c$ and $c+f=d+e$.

So, $a+d+c+f=b+c+d+e$.

Thus, $a+f=b+e$.

So, $(a,b)\sim(e,f)$.

$\blacksquare$



(d)

$$\overline{(1,1)} = \{(1,1),(2,2),(3,3),(6,6),(10,10),\ldots\}$$

$$\overline{(1,2)} = \{(1,2),(2,3),(3,4),(4,5),(20,21),\ldots\}$$

$$\overline{(5,12)} = \{(5,12),(6,13),(7,14),(20,27),(10,17),\ldots\}$$

$$\mathbb{Q} = \left\{ \frac{x}{y} \;\middle|\; \begin{array}{l} x, y \in \mathbb{Z} \\ y \neq 0 \end{array} \right\}$$

Well-defined operations

Let's say one day you and your friend say "Hey let's make a new operation on $\mathbb{Q}$!" Your friend says "Totally! I concur!" You say "What about this operation $\frac{a}{b} \oplus \frac{c}{d} = \frac{a+c}{b+d}$"

new operation symbol

And then joyfully you proceeded to calculate some calculations.

$$\frac{2}{7} \oplus \frac{1}{5} = \frac{2+1}{7+5} = \frac{3}{12}$$

$$\frac{3}{11} \oplus \frac{49}{-1} = \frac{3+49}{11-1} = \frac{52}{10}$$

equals sign

On

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$$\frac{2}{1} \oplus \frac{5}{-1} = \frac{2+5}{1-1} = \frac{7}{0}$$

And then you're like "uh oh!"

One issue is that $\frac{7}{0}$ is not in $\mathbb{Q}$

No good

equals sign

$$\begin{array}{ccccc} \frac{2}{7} & \oplus & \frac{1}{5} & = & \frac{3}{12} \\ \parallel & & \parallel & & \\ \frac{4}{14} & \oplus & \frac{1}{5} & = & \frac{4+1}{14+5} = \frac{5}{19} \end{array}$$

Not equal even though $\frac{2}{7} = \frac{4}{14}$

No good

We say that $\oplus$ is NOT well-defined

Def: Let S be a set. An operation $\oplus$ on S is **well-defined** if

S is closed under $\oplus$

1. For every $x, y \in S$ we have $x \oplus y \in S$.
2. If some or all of the elements of S can be expressed in more than one way, we must show the following:

For every $a, b, c, d \in S$, if $a = b$ and $c = d$ then $a \oplus c = b \oplus d$

Let's make an operation on $\mathbb{Z}_n$.

Given $\overline{a}, \overline{b} \in \mathbb{Z}_n$ define

$$\overline{a} + \overline{b} = \overline{a+b}$$

and

$$\overline{a} \cdot \overline{b} = \overline{ab}$$

[We can make this def. since $a, b \in \mathbb{Z}$ and we know how to compute $a+b$ and ab]

<u>Ex:</u> $\mathbb{Z}_7 = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}\}$

$$\overline{5} + \overline{0} = \overline{5+0} = \overline{5}$$

$$\overline{4} + \overline{6} = \overline{4+6} = \overline{10} = \overline{3}$$

$$\boxed{\begin{array}{l} 10 \equiv 3 \pmod 7 \\ 7 \mid (10-3) \end{array}}$$

$$\overline{3} \cdot \overline{6} = \overline{3 \cdot 6} = \overline{18} = \overline{4}$$

$$\boxed{18 \equiv 4 \pmod 7}$$

$$\overline{4} + \overline{6} = \overline{3}$$

$$\begin{array}{ccc} \overline{4} & + & \overline{6} \\ \| & & \| \\ \overline{18} & + & \overline{-1} \end{array} = \overline{18-1} = \overline{17} = \overline{3}$$

The same!

$\overline{4} = \overline{18}$ and $\overline{6} = \overline{-1}$

$\overline{4} + \overline{6} = \overline{18} + \overline{-1}$

Theorem: Let $n \in \mathbb{Z}$ with $n \geq 2$.
The following operations on $\mathbb{Z}_n$ are well-defined.
Given $\overline{a}, \overline{b} \in \mathbb{Z}_n$ define

$$\overline{a} + \overline{b} = \overline{a+b}$$

and

$$\overline{a} \cdot \overline{b} = \overline{ab}$$

Proof:

1. Given $\overline{a}, \overline{b} \in \mathbb{Z}_n$ with $a, b \in \mathbb{Z}$
   we know that $a+b \in \mathbb{Z}$ and $ab \in \mathbb{Z}$.
   So, $\overline{a+b} \in \mathbb{Z}_n$ and $\overline{ab} \in \mathbb{Z}_n$.
   That is $\overline{a} + \overline{b} \in \mathbb{Z}_n$ and $\overline{a}\,\overline{b} \in \mathbb{Z}_n$.

(2) Let $a, b, c, d \in \mathbb{Z}$ with $\overline{a} = \overline{b}$ and $\overline{c} = \overline{d}$ in $\mathbb{Z}_n$. We need to show that

$$\overline{a} + \overline{c} = \overline{b} + \overline{d} \text{ and } \overline{a}\,\overline{c} = \overline{b}\,\overline{d}.$$

Since $\overline{a} = \overline{b}$ we know $a \equiv b \pmod{n}$.

Since $\overline{c} = \overline{d}$ we know $c \equiv d \pmod{n}$.

Super-duper equivalence class thm

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From last class at the very end we proved that this implies that

$$(a+c) \equiv (b+d) \pmod{n}$$

and $$ac \equiv bd \pmod{n}.$$

Thus, $\overline{a+c} = \overline{b+d}$

and $\overline{ac} = \overline{bd}$.

So, $\overline{a} + \overline{c} = \overline{b} + \overline{d}$

and $\overline{a}\,\overline{c} = \overline{b}\,\overline{d}$. $\blacksquare$