Weds
9/11

Hw 2

16 Let A, B, C, and D be sets.

Prove: $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

proof:

We have that $(x,y) \in (A \times B) \cap (C \times D)$

iff $(x,y) \in A \times B$ and $(x,y) \in C \times D$

iff $x \in A$ and $y \in B$ and $x \in C$ and $y \in D$

iff $x \in A \cap C$ and $y \in B \cap D$

iff $(x,y) \in (A \cap C) \times (B \cap D)$.

$\blacksquare$

$\in D$

Recall from last time

$S = \{1,2,3\}$

$\sim\, = \{(1,1),(2,2),(3,3),(1,3),(3,1)\}$

$\sim$ is an equivalence relation on S.

equivalence classes

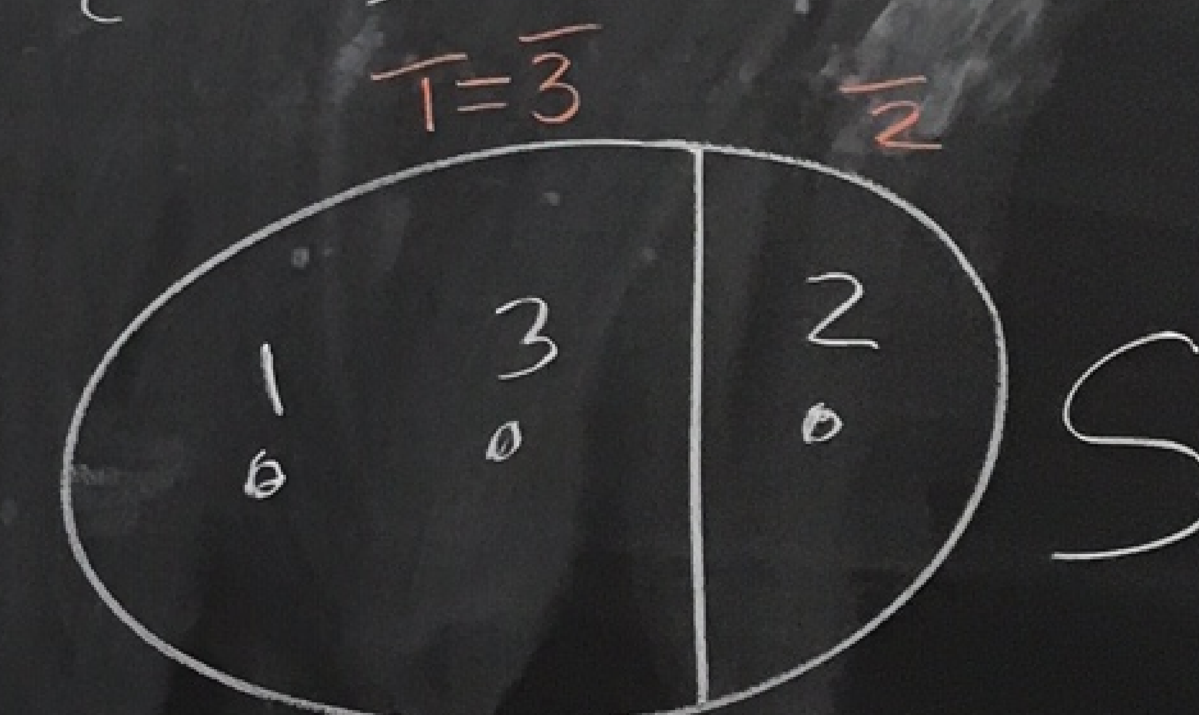
$\bar{1} = \{y \in S \mid 1 \sim y\} = \{1,3\}$

$\bar{2} = \{y \in S \mid 2 \sim y\} = \{2\}$

$\bar{3} = \{y \in S \mid 3 \sim y\} = \{3,1\}$

equal

$S/\!\sim\; = \{\bar{1}, \bar{2}\}$



Super-duper Equivalence relation Theorem

Let $\sim$ be an equivalence relation on a set S.

Let $x, y \in S$. Then:

1. $x \in \overline{x}$
2. $\overline{x} = \overline{y}$ iff $x \in \overline{y}$.
3. $\overline{x} = \overline{y}$ iff $x \sim y$.
4. $\overline{x} \cap \overline{y} = \emptyset$ iff $x \not\sim y$.

Examples from previous example

1. $2 \in \overline{2}$
2. $\overline{1} = \overline{3}$ and $1 \in \overline{3}$ (also $3 \in \overline{1}$)
3. $\overline{1} = \overline{3}$ and $1 \sim 3$.
4. $\overline{1} \cap \overline{2} = \{1,3\} \cap \{2\} = \emptyset$ and $1 \not\sim 2$.

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Note (3) and (4) imply that either
$\overline{x} = \overline{y}$ OR $\overline{x} \cap \overline{y} = \phi$.

<u>Proof</u>: Let $x, y \in S$.

(1) Recall that

$\overline{x} = \{a \in S \mid x \sim a\}$.

Since $\sim$ is reflexive we know that $x \sim x$.

So, $x \in \overline{x}$.

(2) ($\Rightarrow$) Assume that $\overline{x} = \overline{y}$.

We want to show that $x \in \overline{y}$.

By part 1, we know $x \in \overline{x}$.

Since $\overline{x} = \overline{y}$ we get that $x \in \overline{y}$.

($\Leftarrow$) Assume that $x \in \overline{y}$.

We need to show that $\overline{x} = \overline{y}$.

$\boxed{\overline{x} \subseteq \overline{y}}$: Let $z \in \overline{x} = \{a \in S \mid x \sim a\}$

So, $x \sim z$.

Since $x \in \overline{y} = \{a \in S \mid y \sim a\}$

we have $y \sim x$.

Since $y \sim x$ and $x \sim z$
and $\sim$ is an equivalence relation we have transitivity,
so $y \sim z$.

Thus, $z \in \overline{y}$. So, $\overline{x} \subseteq \overline{y}$.

$\boxed{\overline{y} \subseteq \overline{x}}$: Let $z \in \overline{y}$.

Then, $y \sim z$.

Since $x \in \overline{y}$ we have $y \sim x$.

Since $\sim$ is symmetric
and $y \sim x$ we have
that $x \sim y$.

Since $x \sim y$ and $y \sim z$
by transitivity
we have $x \sim z$.

So, $z \in \overline{x}$.

Thus, $\overline{y} \subseteq \overline{x}$.

$\sim x.$

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Since $\overline{x} \subseteq \overline{y}$ and $\overline{y} \subseteq \overline{x}$ we have that $\overline{x} = \overline{y}$.

③ By part 2,
$\overline{x} = \overline{y}$ iff $x \in \overline{y}$.

Note $x \in \overline{y}$ iff $y \sim x$.
Also since $\sim$ is symmetric
$y \sim x$ iff $x \sim y$.

Thus, $\overline{x} = \overline{y}$ iff $x \in \overline{y}$
iff $y \sim x$
iff $x \sim y$.

④ Instead of proving
"$\overline{x} \cap \overline{y} = \phi$ iff $x \not\sim y$"
we prove
"$\overline{x} \cap \overline{y} \neq \phi$ iff $x \sim y$"

P	Q	$\neg P$	$\neg Q$	P iff Q	$(\neg P)$ iff $(\neg Q)$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	F	F
F	F	T	T	T	T

$\boxed{\begin{array}{l} P \text{ iff } Q \\ \text{is equivalent to} \\ (\neg P) \text{ iff } (\neg Q) \end{array}}$

($\Leftarrow$) Suppose $x \sim y$.
Then by part 2, $\overline{x} = \overline{y}$.
By part 1, $x \in \overline{x}$.
Since $\overline{x} = \overline{y}$ we have $\overline{x} \cap \overline{y} = \overline{x}$.
So, $x \in \overline{x} \cap \overline{y}$.
Thus, $\overline{x} \cap \overline{y} \neq \phi$.

($\Rightarrow$) Assume that $\overline{x} \cap \overline{y} \neq \phi$.
So, there exists $z \in S$
with $z \in \overline{x} \cap \overline{y}$.

So, $z \in \overline{x}$ and
$z \in \overline{y}$.
Thus, $x \sim z$ and
$y \sim z$.
By the symmetric-ness
of $\sim$ we get
that $z \sim y$.
By transitivity-ness
of $\sim$ since $x \sim z$ and
$z \sim y$ we get $x \sim y$.

$\boxed{\text{DONE}}$

HW 2

$\boxed{\text{Recall: } \mathcal{P}(X) = \{Y \mid Y \subseteq X\}}$

(14) (a) $\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B)$.

proof: $Y \in \mathcal{P}(A \cap B)$

iff $Y \subseteq A \cap B$

iff $Y \subseteq A$ and $Y \subseteq B$

iff $Y \in \mathcal{P}(A)$ and $Y \in \mathcal{P}(B)$.

iff $Y \in \mathcal{P}(A) \cap \mathcal{P}(B)$. $\boxed{\text{FIN}}$

Lemma: Let Y, A, and B be sets. Then, $Y \subseteq A \cap B$ iff $Y \subseteq A$ and $Y \subseteq B$.

proof:

($\Rightarrow$) Suppose $Y \subseteq A \cap B$.

Let $y \in Y$.

Then $y \in A \cap B$.

So, $y \in A$ and $y \in B$.

Thus, $Y \subseteq A$ and $Y \subseteq B$.

($\Leftarrow$) Suppose $Y \subseteq A$ and $Y \subseteq B$.

Let $y \in Y$.

Then from above $y \in A$ and $y \in B$.

So, $y \in A \cap B$.

Thus, $Y \subseteq A \cap B$.

Kemma