Mon
12/9

14.1

#9 Show that $\{0,1\} \times \mathbb{N}$ has the same cardinality as $\mathbb{N}$ by constructing a bijection between the two sets.

a_2 .

$n_2 - a_2 = 2n_2$
be even.

$\{0,1\} \times \mathbb{N} \xrightarrow{f} \mathbb{N}$

$$
\begin{aligned}
(1,1) &\to 1\\
(0,1) &\to 2\\
(1,2) &\to 3\\
(0,2) &\to 4\\
(1,3) &\to 5\\
(0,3) &\to 6\\
(1,4) &\to 7\\
(0,4) &\to 8\\
\vdots &\quad \vdots
\end{aligned}
$$

$$f(a,n) = \begin{cases} 2n, & \text{if } a = 0 \\ 2n-1, & \text{if } a = 1 \end{cases}$$

$$f(a,n) = 2n - a$$

<u>f is a bijection</u>

① <u>f is one-to-one</u>

Suppose $f(a_1, n_1) = f(a_2, n_2)$ for some $a_1, a_2 \in \{0,1\}$ and $n_1, n_2 \in \mathbb{N}$.

Since $f(a_1,n_1)=f(a_2,n_2)$ we have $2n_1-a_1=2n_2-a_2$.

case 1: Suppose $a_1=0$.

Then, $2n_2-a_2=2n_1-a_1=2n_1$

So, $2n_2-a_2$ is even.

Since $a_2=0$ or $a_2=1$ we must have $a_2=0$ to have $2n_2-a_2=2n_2$ be even.

case 2: Suppose $a_1=1$.

Then, $2n_2-a_2=2n_1-a_1=2n_1-1$ is odd.

So, $a_2\neq 0$, because then $2n_2-a_2=2n_2$ would be even.

So, $a_2=1$.

n_2

en.

So, if $a_1 = 0$ then $a_2 = 0$, and if $a_1 = 1$ then $a_2 = 1$.

So, $a_1 = a_2$.

$f(a,n) = 2n - a$

Since $2n_1 - a_1 = 2n_2 - a_2$ and $a_1 = a_2$ we get $2n_1 = 2n_2$.

Thus, $n_1 = n_2$.

Therefore, $(a_1, n_1) = (a_2, n_2)$.

So, f is 1-1.

<u>f is onto</u>

Let $b \in \mathbb{N}$.

<u>case 1: Suppose b is even.</u>

Then $b = 2n$ for some $n \in \mathbb{N}$.

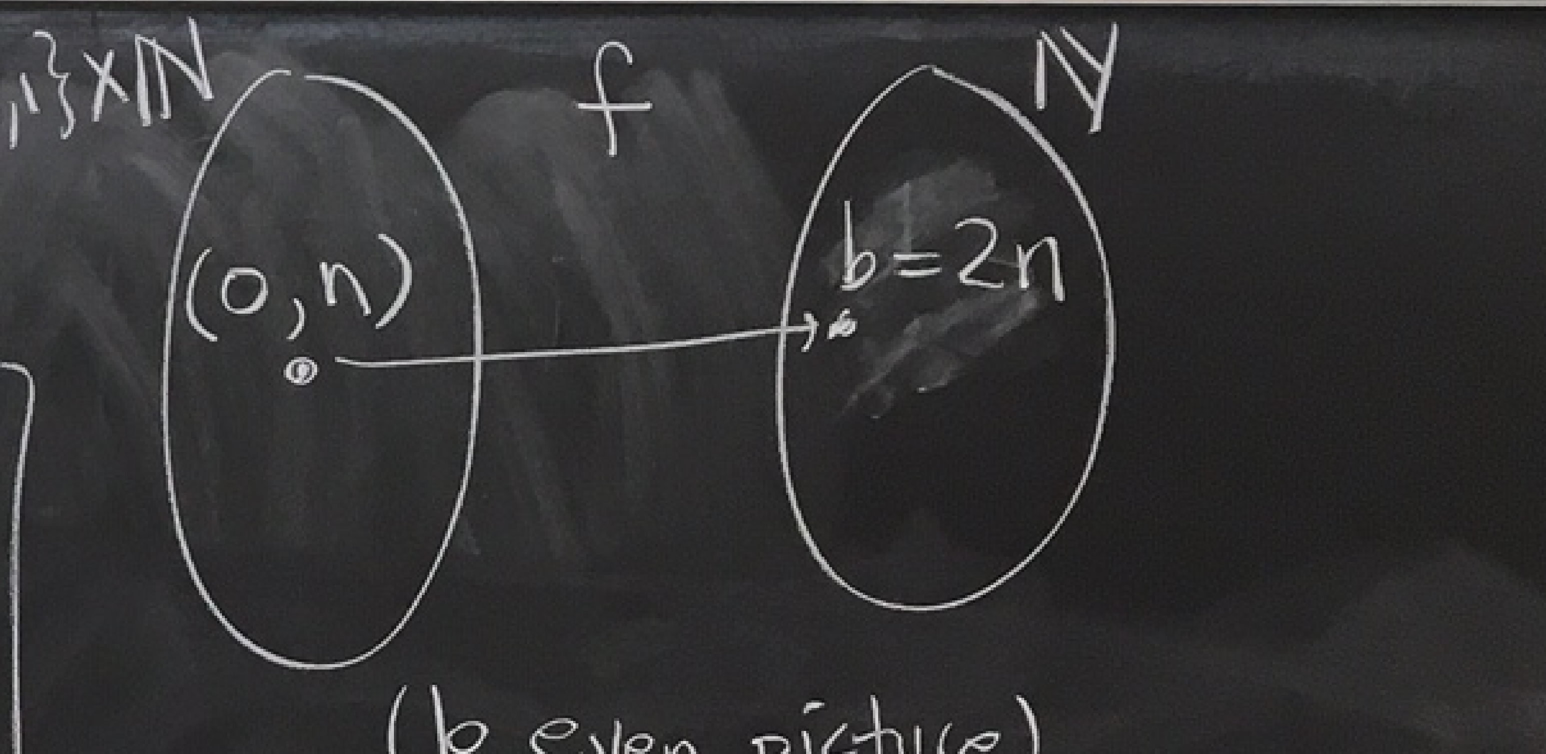
And $f(0,n) = 2n - 0 = b$.

<u>case 2: Suppose b is odd.</u>

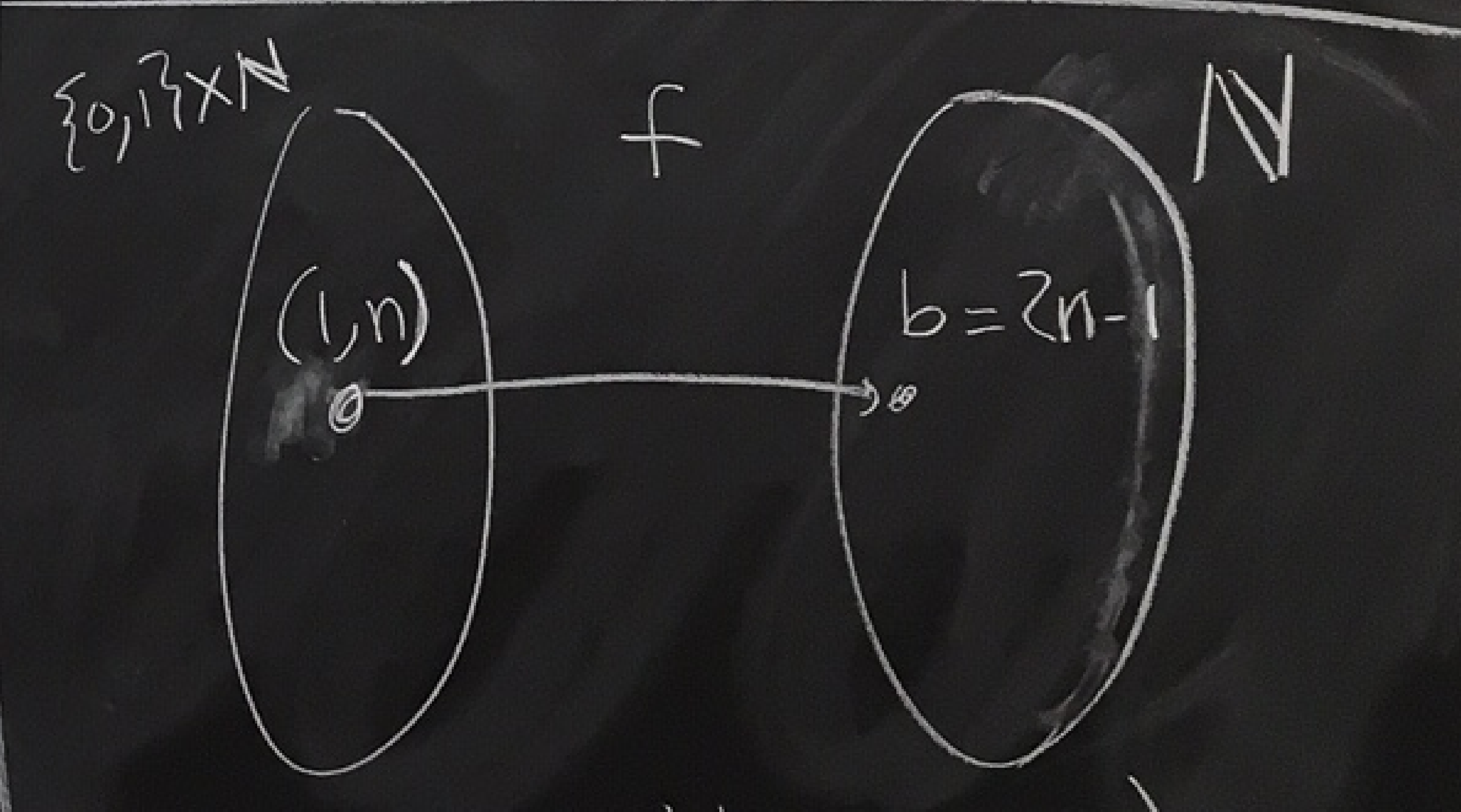
Then $b = 2n - 1$ for some $n \in \mathbb{N}$.

And $f(1,n) = 2n - 1 = b$.

Therefore f is onto. $\blacksquare$



(b even picture)



(b is odd picture)

$$\boxed{\begin{aligned} f: A &\longrightarrow B \\ 3k &\longmapsto 5k \end{aligned}}$$

14.1

⑤ Same question with

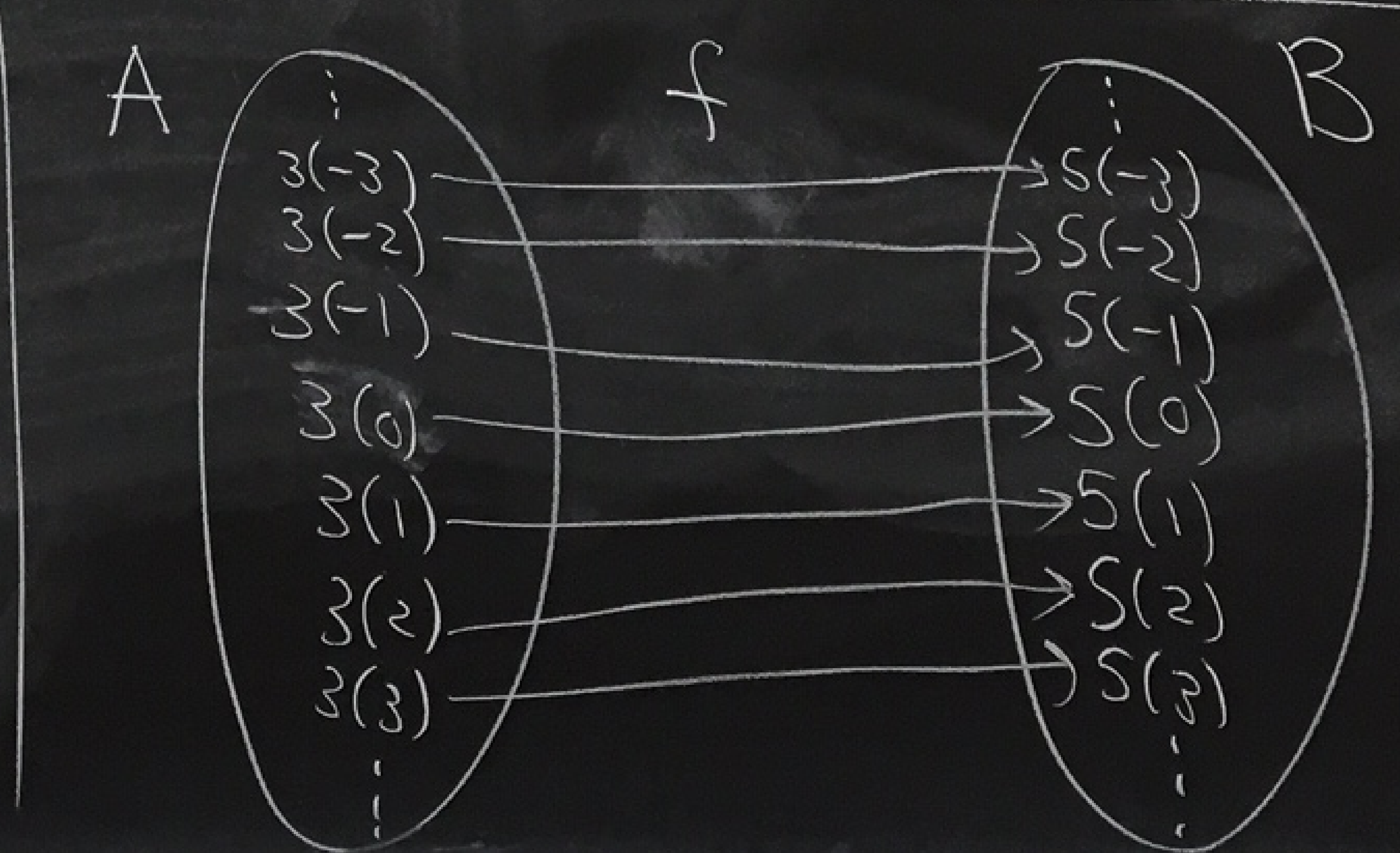
$$A = \{3k \mid k \in \mathbb{Z}\} = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$$

and $B = \{5k \mid k \in \mathbb{Z}\} = \{\ldots, -10, -5, 0, 5, 10, \ldots\}$

<u>scratchwork</u>

$$3k \longmapsto 5k$$

$$x \longmapsto \frac{5}{3}x$$



Define $f: A \to B$ by $f(x) = \frac{5}{3}x$

f is well-defined

Let $x \in A$.

Then $x = 3k$ for some $k \in \mathbb{Z}$.

And $f(x) = \frac{5}{3}x = \frac{5}{3}(3k)$

$= 5k \in B$.

f is 1-1

Suppose $f(x_1) = f(x_2)$ where $x_1, x_2 \in A$.

Then, $\frac{5}{3}x_1 = \frac{5}{3}x_2$.

Cancelling the $\frac{5}{3}$ gives $x_1 = x_2$.

f is onto

Let $b \in B$.

Then,

$b = 5k$

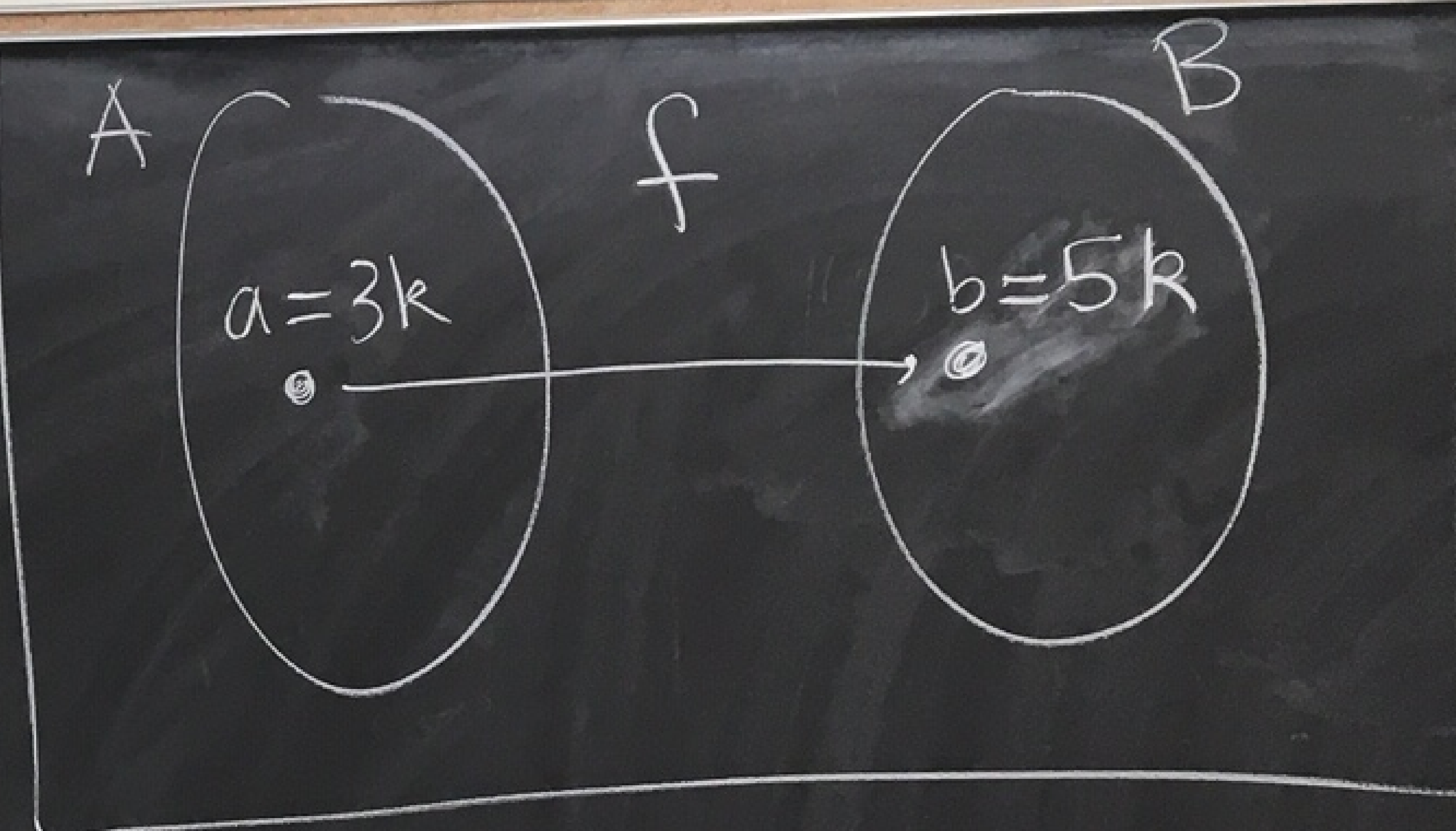
for some $k \in \mathbb{Z}$.

Consider $a = 3k$.

Then $a \in A$ and

$f(a) = f(3k) = \frac{5}{3}(3k) = 5k = b$.

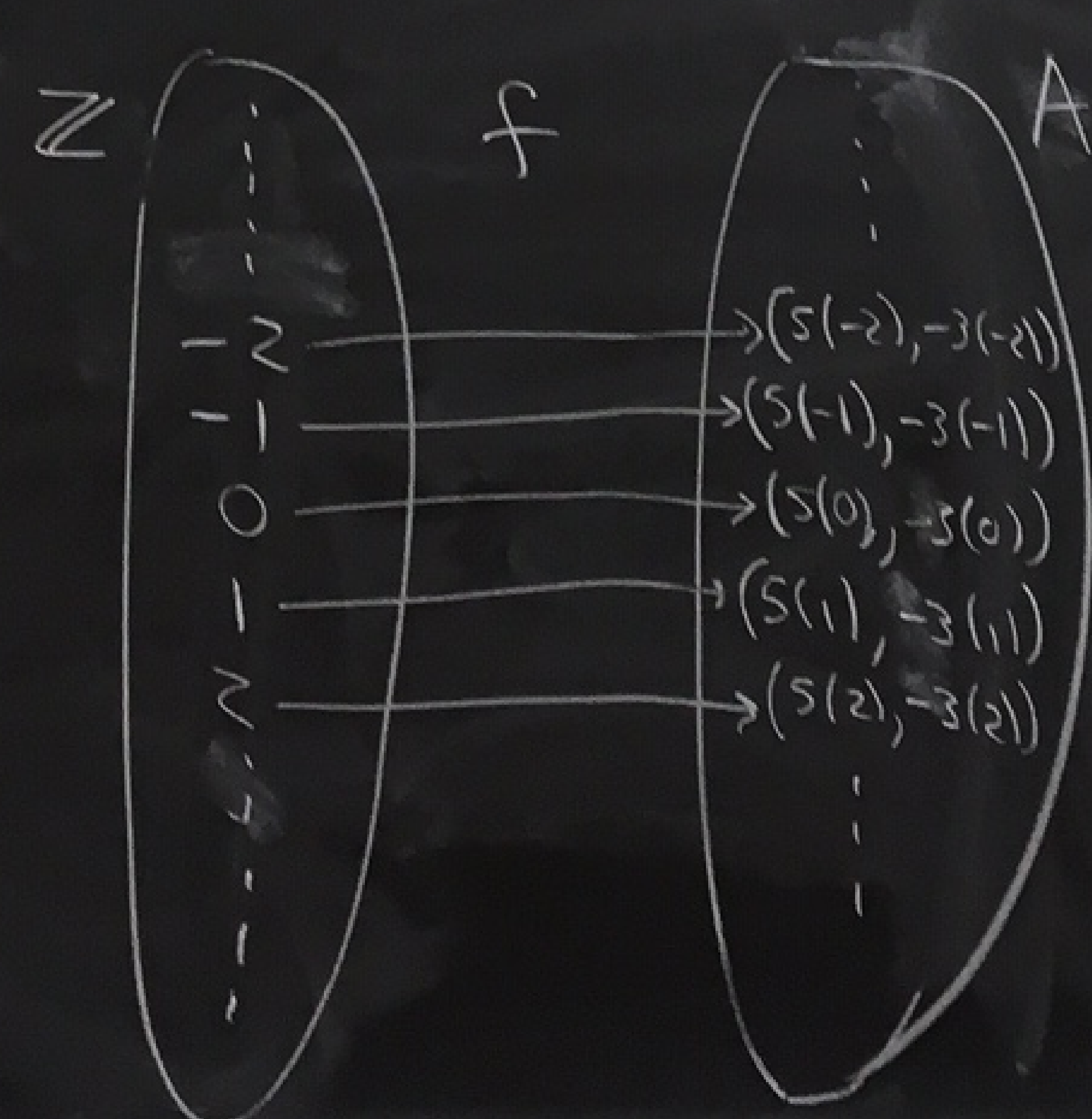
So, f is onto. ∎



14.2

③ Prove that $A=\{(5n,-3n) \mid n\in\mathbb{Z}\}=\{\ldots,\underset{(-10,6)}{(5(-2),-3(-2))},\underset{(-5,3)}{(5(-1),-3(-1))},\underset{(0,0)}{(5(0),-3(0))},\underset{(5,-3)}{(5(1),-3(1))},\ldots\}$ is countably infinite.

Define $f:\mathbb{Z}\longrightarrow A$ by $f(n)=(5n,-3n)$



f is 1-1

Suppose $f(n_1)=f(n_2)$ where $n_1,n_2\in\mathbb{Z}$

Then $(5n_1,-3n_1)=(5n_2,-3n_2)$.

So, $5n_1=5n_2$ and $-3n_1=-3n_2$.

Dividing $5n_1=5n_2$ by 5 gives $n_1=n_2$.

So, f is 1-1.

f

Let

Then

And

f

f

So,

The

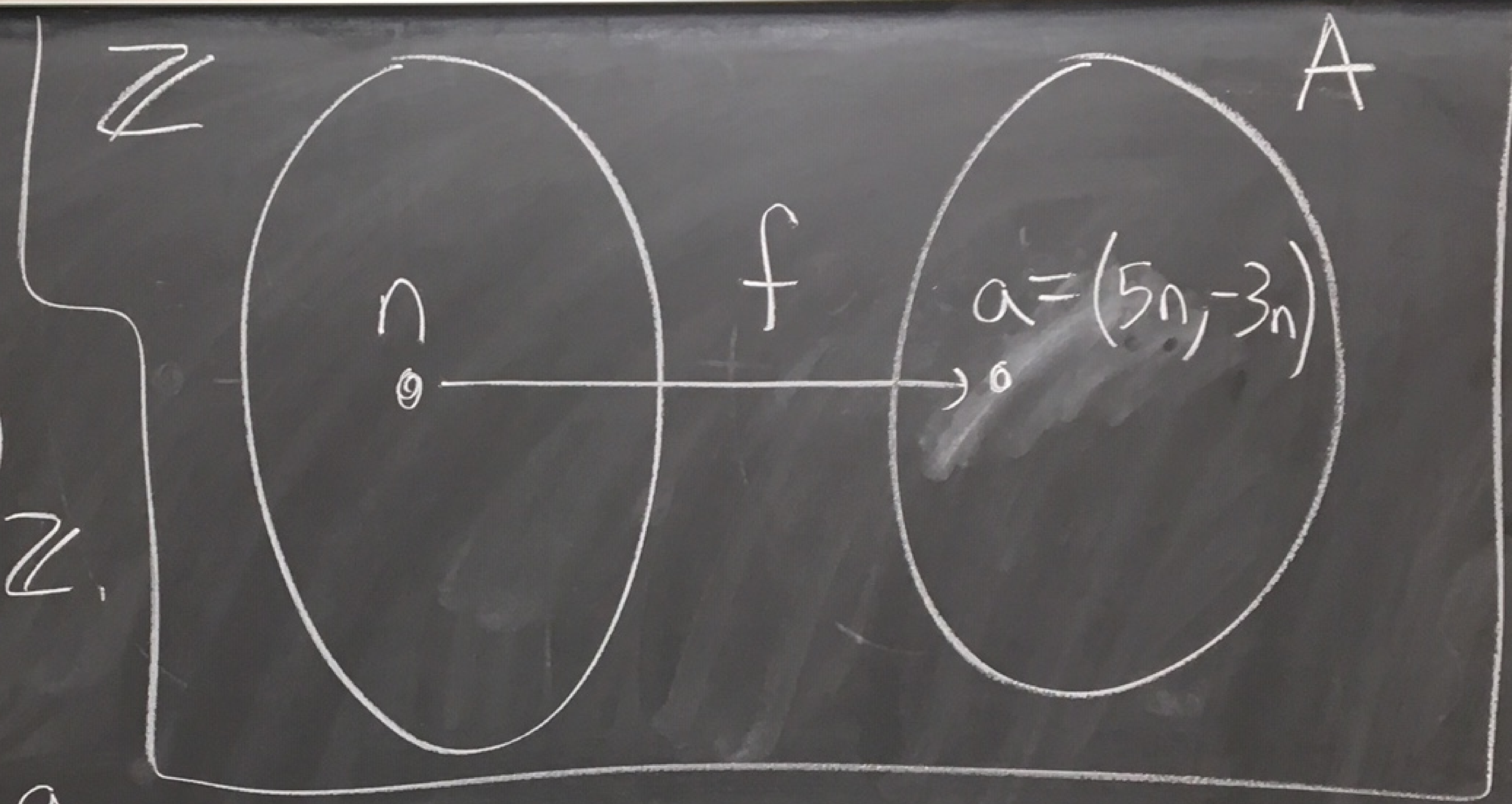
f is onto

Let $a \in A$.

Then $a = (5n, -3n)$ for some $n \in \mathbb{Z}$.

And,
$f(n) = (5n, -3n) = a$.

So, f is onto.



Therefore, f is a bijection. So, $|\mathbb{Z}| = |A|$.

Thus, $|\mathbb{N}| = |\mathbb{Z}| = |A|$.

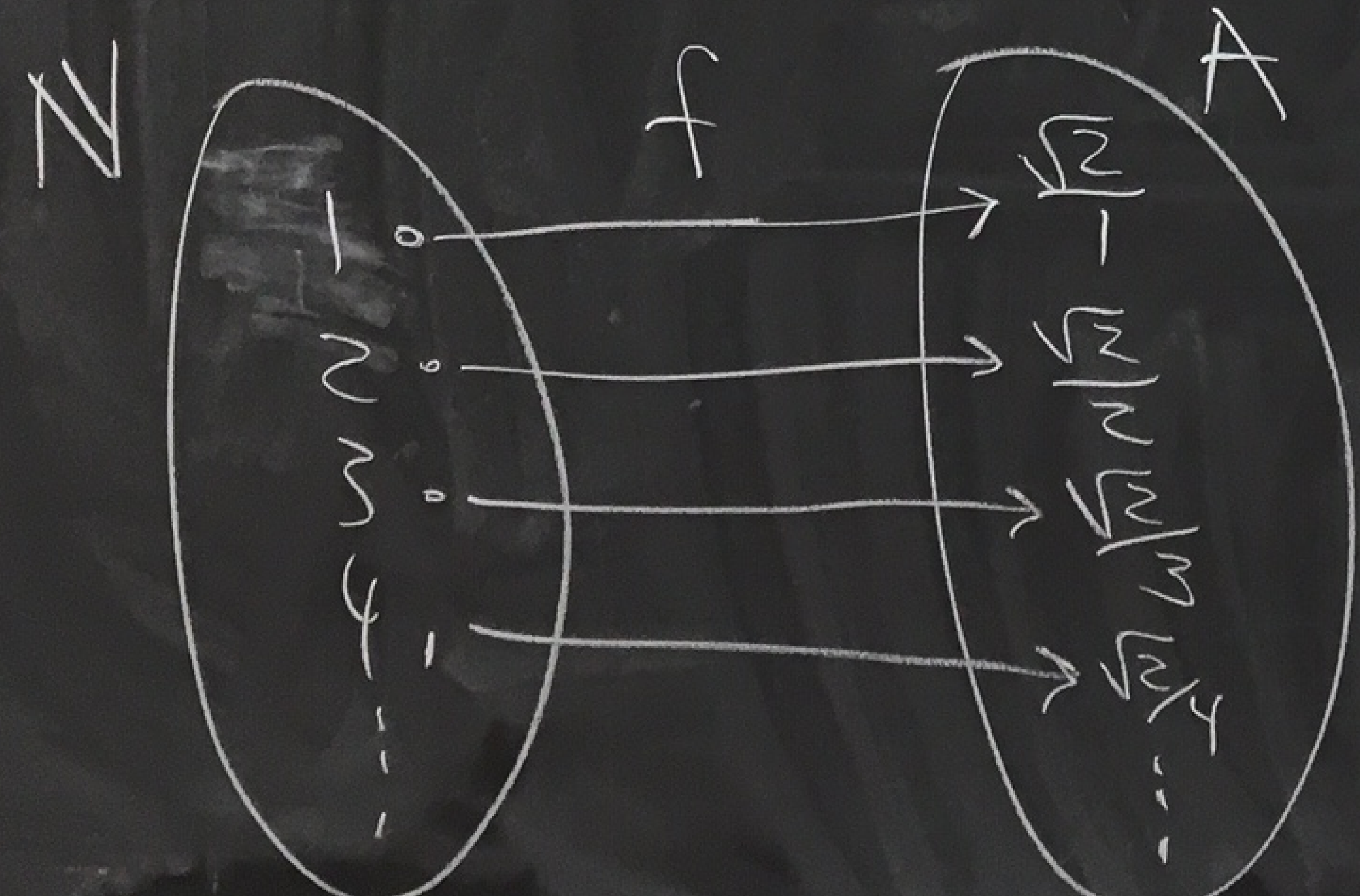
So, A is infinitely countable.

$\blacksquare$

14.2

(10) $A = \left\{ \frac{\sqrt{2}}{n} \mid n \in \mathbb{N} \right\} = \left\{ \frac{\sqrt{2}}{1}, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{4}, \ldots \right\}$

Is A countably infinite?



Define $f : \mathbb{N} \to A$

by $f(n) = \frac{\sqrt{2}}{n}$

f is 1-1

Suppose $f(n_1)=f(n_2)$ where $n_1, n_2 \in \mathbb{N}$

Then, $\frac{\sqrt{2}}{n_1} = \frac{\sqrt{2}}{n_2}$.

So, $(\sqrt{2})n_2 = (\sqrt{2})n_1$

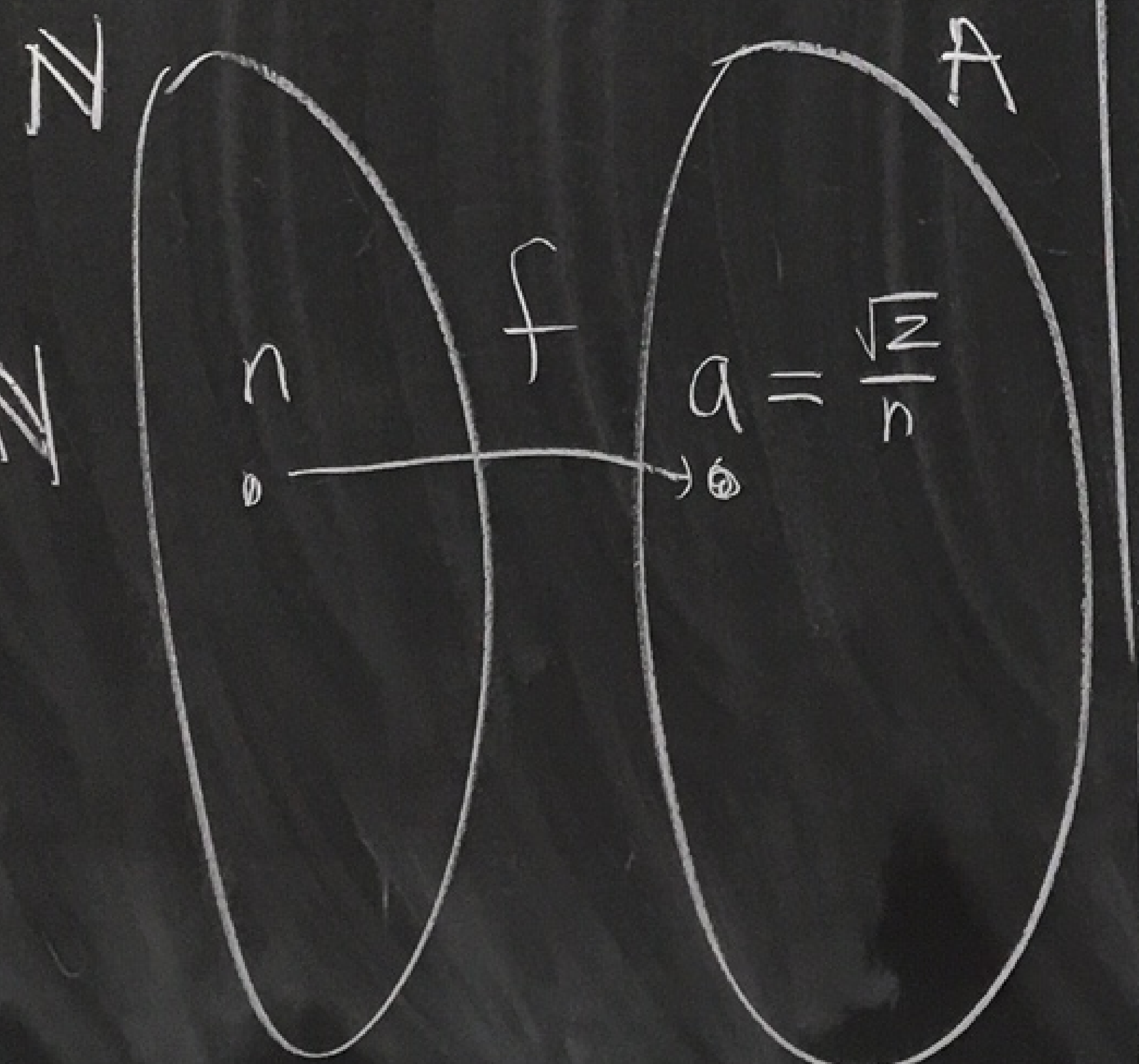
Thus, $n_2 = n_1$.

f is onto

Let $a \in A$

Then, $a = \frac{\sqrt{2}}{n}$ where $n \in \mathbb{N}$

And $f(n) = \frac{\sqrt{2}}{n} = a$.



So, f is a bijection.

Therefore, $|\mathbb{N}| = |A|$

And A is countably infinite