12/4
Weds

Def: Let A and B be sets.

1. $|A| = |B|$ means there exists a bijection $f: A \to B$.

2. $|A| \leq |B|$ means there exists a one-to-one function $f: A \to B$

3. $|A| < |B|$ means there exists a one-to-one function $f: A \to B$, but there is no bijection $g: A \to B$

<u>Ex:</u> $|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$

$|\mathbb{N}| < |\mathbb{R}|$

The next theorem will show that

$$|\mathbb{N}| < |\mathbb{R}| < |\mathcal{P}(\mathbb{R})| < |\mathcal{P}(\mathcal{P}(\mathbb{R}))| < |\mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{R})))| < \cdots$$

So there is no "biggest" set.

Theorem: Let A be a set. Then $|A| < |\mathcal{P}(A)|$.

pf:

If A is finite then $|A| < 2^{|A|} = |\mathcal{P}(A)|$.

Now assume A is infinite.

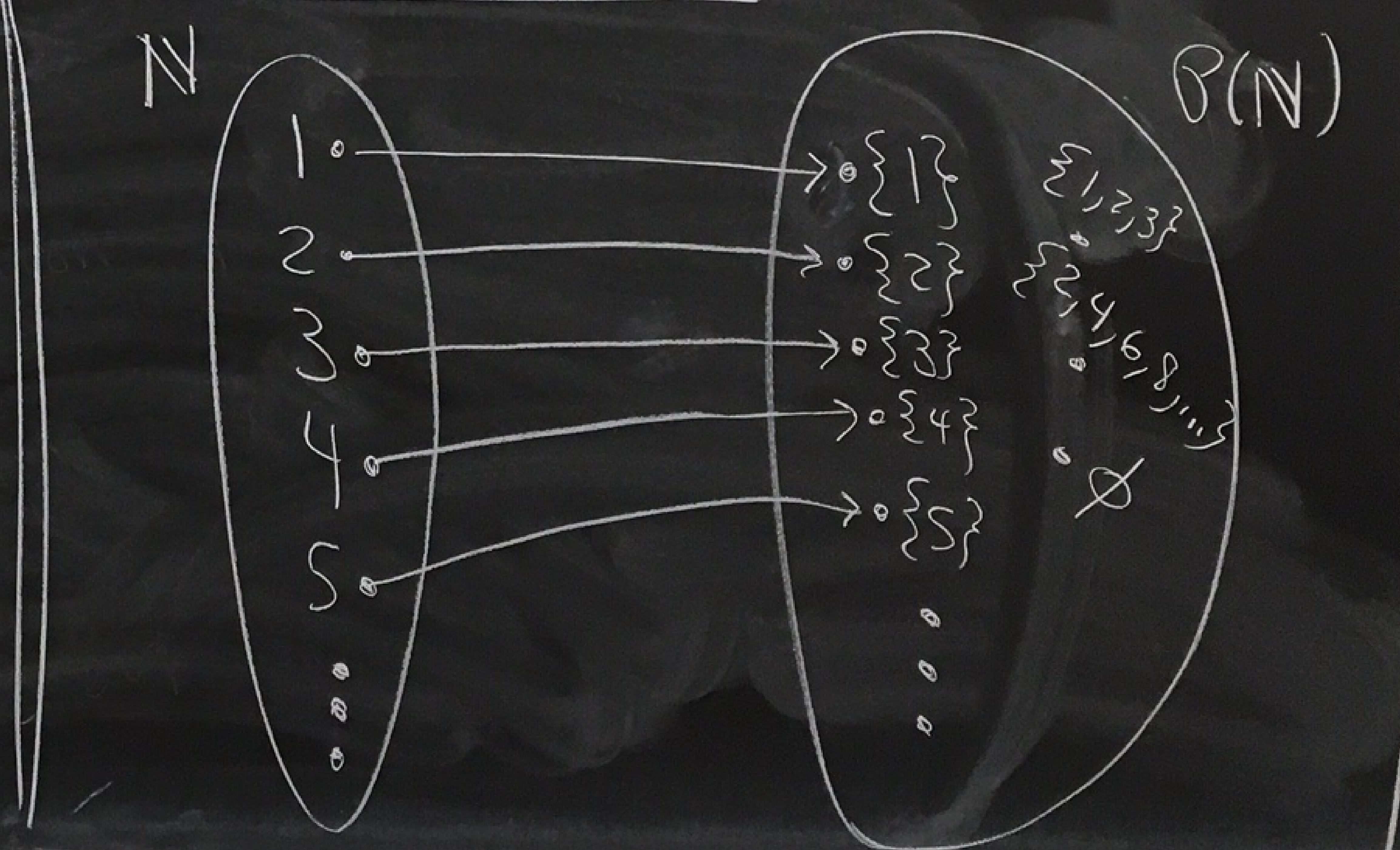
Step 1: Create $f: A \to \mathcal{P}(A)$ that is 1-1

Define $f: A \to \mathcal{P}(A)$ by $f(a) = \{a\}$.

Let's show f is 1-1.

Suppose $f(x) = f(y)$ where $x, y \in A$.

Side example of $f: \mathbb{N} \to \mathcal{P}(\mathbb{N})$ that is 1-1.



Then $\{x\} = \{y\}$. $\leftarrow$ $\boxed{\begin{array}{l} x \in \{y\} \\ \text{So } x = y. \end{array}}$

So, $x = y$.

So, $|A| \leq |\mathcal{P}(A)|$.

Step 2: Show there is no bijection $g: A \to \mathcal{P}(A)$.

Suppose $g: A \to \mathcal{P}(A)$.

We will show g cannot be onto.

We do this by constructing an element $B \in \mathcal{P}(A)$ that isn't in the range of g.

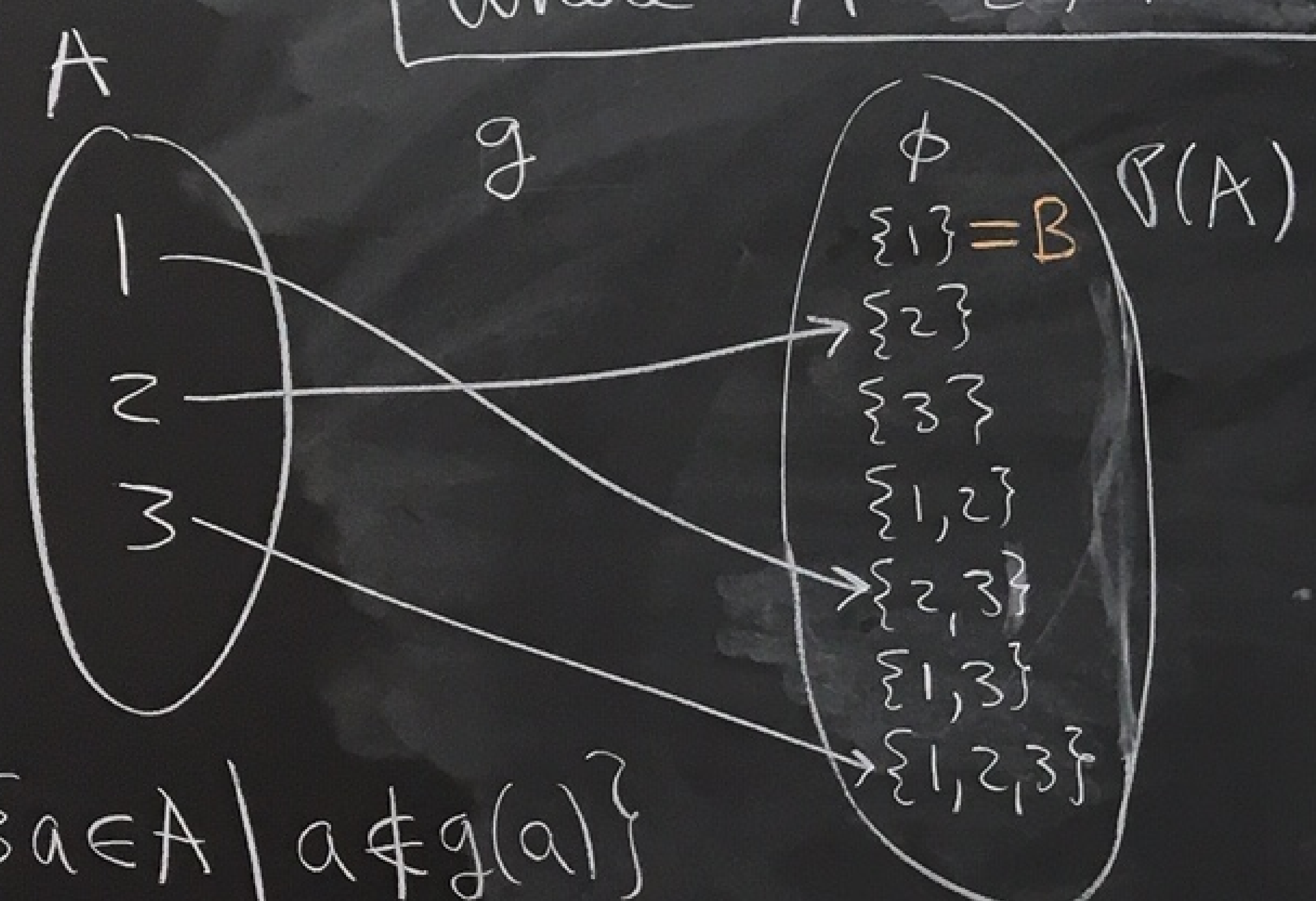
Let

$$B = \{a \in A \mid a \notin g(a)\}.$$

Note $B \subseteq A$, so $B \in \mathcal{P}(A)$.

Let's show $B \notin \text{range}(g)$.

Example of B where $A = \{1,2,3\}$



$B = \{a \in A \mid a \notin g(a)\}$

$1 \notin g(1)$? Yes. $1 \notin \{2,3\}$

$2 \notin g(2)$? No. $2 \in \{2\}$

$3 \notin g(3)$? No, $3 \in \{1,2,3\}$

$B = \{1\}$

$B \notin \text{range}(g)$

(proof continued...)

Let $a \in A$.

We will show $g(a) \neq B$.

One we've shown this then we know there is no element of A that maps to B.

<u>case 1: Suppose $a \in g(a)$.</u>

Thus, by the def. of B, $\boxed{a \notin B.}$

Why can't we have $g(a) = B$?

Suppose $g(a) = B$.

By assumption, $a \in g(a)$. So if $g(a) = B$, then $\boxed{a \in B.}$

B.

$\boxed{a \in B.}$

Then we would have both $a \notin B$ and $a \in B$, which can't happen.

So in this case, $g(a) \neq B$.

Case 2: Suppose $a \notin g(a)$

Since $a \notin g(a)$ we have $\boxed{a \in B.}$

Why can't $g(a) = B$?

If $g(a) = B$, then since $a \notin g(a)$ we would get $\boxed{a \notin B.}$

We can't have $a \in B$ and $a \notin B$.

So in this case, $g(a) \neq B$.

In both cases $g(a) \neq B$.

Since a was an arbitrary element of A, we have shown $B \notin \text{range}(g)$.

So, g isn't onto. $\blacksquare$

Cantor-Bernstein-Schröder Theorem

Let A and B be sets.

If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

That is, if there exist one-to-one functions

$f: A \to B$ and $g: B \to A$, then

there exists a bijection $h: A \to B$.

Thm: $|\mathbb{R}| = |\mathcal{P}(\mathbb{N})|$.

$$\boxed{\begin{array}{ccccccc} |\mathbb{Q}| & & & & & & \\ \shortparallel & & & & & & \\ |\mathbb{N}| & < & |\mathbb{R}| & < & |\mathcal{P}(\mathbb{R})| & < & \ldots, \\ \shortparallel & & \shortparallel & & & & \\ |\mathbb{Z}| & & |\mathcal{P}(\mathbb{N})| & & & & \end{array}}$$

$\boxed{14.1}$

9 Show $\left|\{0,1\} \times \mathbb{N}\right| = |\mathbb{N}|$

$$f(a,n) = \begin{cases} 2n, & \text{if } a = 0 \\ 2n-1, & \text{if } a = 1 \end{cases}$$

$$f(a,n) = 2n - a$$