Monday
12/2

Recall from last time

We say that two sets A and B have the <u>same cardinality</u> if there exists a bijection between them. If so, we write $|A| = |B|$.
If no such bijection exists we write $|A| \neq |B|$.

Ex:

$|\mathbb{N}| = |\mathbb{Z}|$

$|\mathbb{N}| \neq |\mathbb{R}|$

$|(0,1)| = |(0,\infty)|$

Def: Let A be a set.

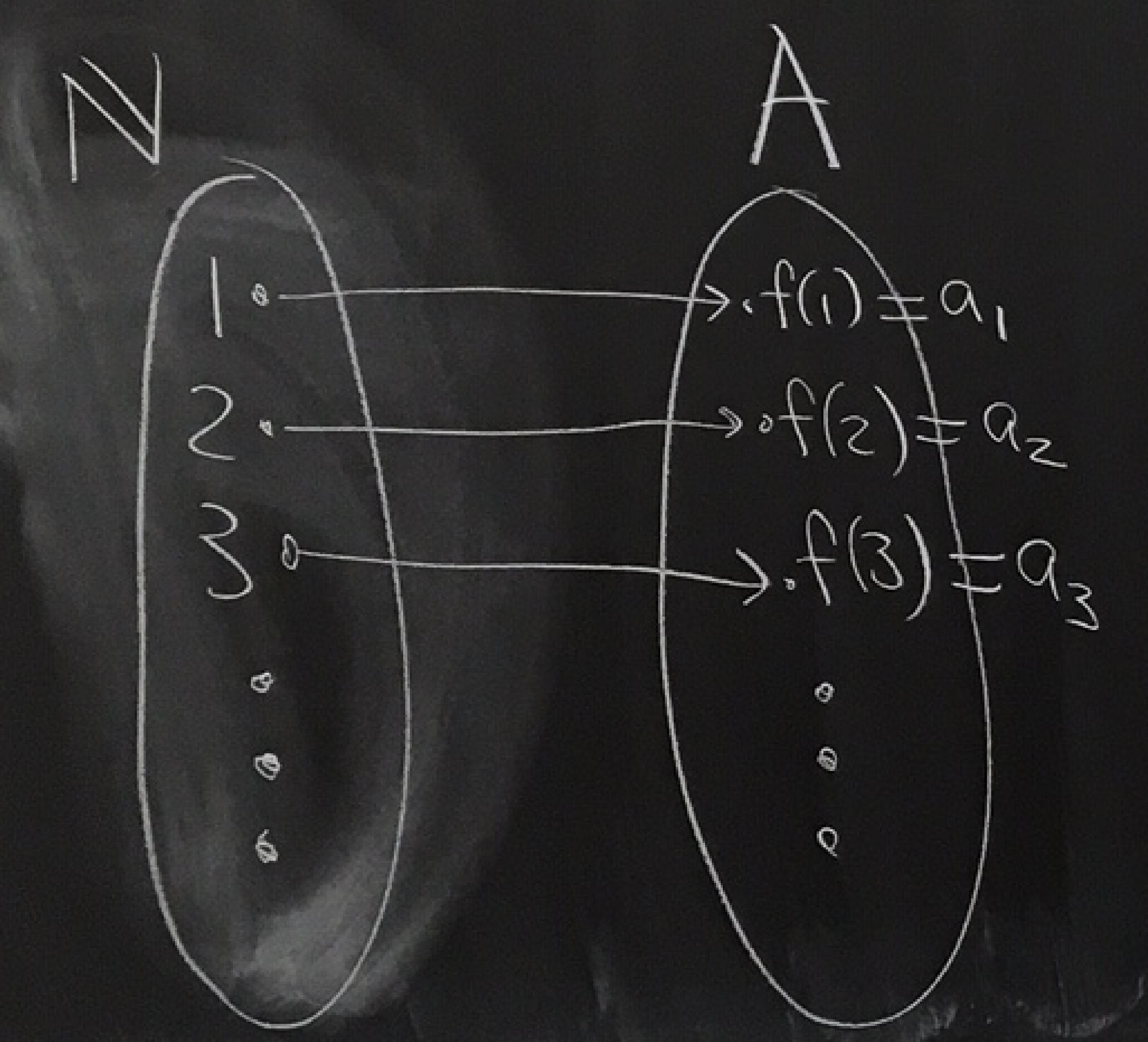
- We say that A is <u>countably infinite</u> if $|A| = |\mathbb{N}|$
- We say that A is <u>countable</u> if either A is finite or A is countably infinite.
- We say that A is <u>uncountable</u> if A is not countable, that is A is infinite and $|A| \neq |\mathbb{N}|$.

Ex: $\mathbb{Z}$ is countably infinite. $\mathbb{Z}$ is countable.

$\{1,2,4\}$ is countable. $\mathbb{R}$ is uncountable.

<u>Theorem:</u> A set A is countably infinite iff its elements can be arranged in an infinite list $a_1, a_2, a_3, \ldots$ with no repeats.

<u>proof:</u>



($\Rightarrow$) Suppose A is countably infinite. Then there exists $f: \mathbb{N} \to A$ that is a bijection. Define $a_i = f(i)$.

Then consider the list $a_1, a_2, a_3, \ldots$

Since f is onto, our list gives all of A.

Since f is 1-1, the list has no repeats.

($\Leftarrow$) Suppose the elements of A can be arranged in an infinite list

$a_1, a_2, a_3, \ldots$ with no repeats.

Define $f: \mathbb{N} \to A$ by $f(i) = a_i$.

Since the list has no repeats, f is 1-1.

Since the list contains all the elements of A, f is onto.

So f is a bijection and $|\mathbb{N}| = |A|$.

So, A is countably infinite. $\blacksquare$

Theorem: $\mathbb{Q}$ is countably infinte.

Start here

0	1	-1	2	-2	3	-3	$\cdots$
$\frac{0}{1}$	$\frac{1}{1}$	$\frac{-1}{1}$	$\frac{2}{1}$	$\frac{-2}{1}$	$\frac{3}{1}$	$\frac{-3}{1}$	$\cdots$
$\frac{1}{2}$	$\frac{-1}{2}$	$\cancel{\frac{2}{2}}$	$\cancel{\frac{-2}{2}}$	$\frac{3}{2}$	$\frac{-3}{2}$	$\cdots$
$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{3}$	$\frac{-2}{3}$	$\cancel{\frac{3}{3}}$	$\cancel{\frac{-3}{3}}$	$\cdots$
$\frac{1}{4}$	$\frac{-1}{4}$	$\cancel{\frac{2}{4}}$	$\cancel{\frac{-2}{4}}$	$\frac{3}{4}$	$\frac{-3}{4}$	$\cdots$
$\frac{1}{5}$	$\frac{-1}{5}$	$\frac{2}{5}$	$\frac{-2}{5}$	$\frac{3}{5}$	$\frac{-3}{5}$	$\cdots$
$\frac{1}{6}$	$\frac{-1}{6}$	$\cancel{\frac{2}{6}}$	$\cancel{\frac{-2}{6}}$	$\cancel{\frac{3}{6}}$	$\cancel{\frac{-3}{6}}$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

0	1	−1	2	−2	3	−3	4	−4	5	−5	⋯
$\frac{0}{1}$	$\frac{1}{1}$	$\frac{-1}{1}$	$\frac{2}{1}$	$\frac{-2}{1}$	$\frac{3}{1}$	$\frac{-3}{1}$	$\frac{4}{1}$	$\frac{-4}{1}$	$\frac{5}{1}$	$\frac{-5}{1}$	⋯
	$\frac{1}{2}$	$\frac{-1}{2}$	$\frac{2}{3}$	$\frac{-2}{3}$	$\frac{3}{2}$	$\frac{-3}{2}$	$\frac{4}{3}$	$\frac{-4}{3}$	$\frac{5}{2}$	$\frac{-5}{2}$	⋯
	$\frac{1}{3}$	$\frac{-1}{3}$	$\frac{2}{5}$	$\frac{-2}{5}$	$\frac{3}{4}$	$\frac{-3}{4}$	$\frac{4}{5}$	$\frac{-4}{5}$	$\frac{5}{3}$	$\frac{-5}{3}$	⋯
	$\frac{1}{4}$	$\frac{-1}{4}$	$\frac{2}{7}$	$\frac{-2}{7}$	$\frac{3}{5}$	$\frac{-3}{5}$	$\frac{4}{7}$	$\frac{-4}{7}$	$\frac{5}{4}$	$\frac{-5}{4}$	⋯
	$\frac{1}{5}$	$\frac{-1}{5}$	$\frac{2}{9}$	$\frac{-2}{9}$	$\frac{3}{7}$	$\frac{-3}{7}$	$\frac{4}{9}$	$\frac{-4}{9}$	$\frac{5}{6}$	$\frac{-5}{6}$	⋯
	$\frac{1}{6}$	$\frac{-1}{6}$	$\frac{2}{11}$	$\frac{-2}{11}$	$\frac{3}{8}$	$\frac{-3}{8}$	$\frac{4}{11}$	$\frac{-4}{11}$	$\frac{5}{7}$	$\frac{-5}{7}$	⋯
	$\frac{1}{7}$	$\frac{-1}{7}$	$\frac{2}{13}$	$\frac{-2}{13}$	$\frac{3}{10}$	$\frac{-3}{10}$	$\frac{4}{13}$	$\frac{-4}{13}$	$\frac{5}{8}$	$\frac{-5}{8}$	⋯
	$\frac{1}{8}$	$\frac{-1}{8}$	$\frac{2}{15}$	$\frac{-2}{15}$	$\frac{3}{11}$	$\frac{-3}{11}$	$\frac{4}{15}$	$\frac{-4}{15}$	$\frac{5}{9}$	$\frac{-5}{9}$	⋯
	$\frac{1}{9}$	$\frac{-1}{9}$	$\frac{2}{17}$	$\frac{-2}{17}$	$\frac{3}{13}$	$\frac{-3}{13}$	$\frac{4}{17}$	$\frac{-4}{17}$	$\frac{5}{11}$	$\frac{-5}{11}$	⋯
	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	

We put $\mathbb{Q}$ into an infinite list with no repeats by weaving through the infinite table as indicated and removing repeats as we encounter them. Here are the first entries in the list:

$$0, 1, \frac{1}{2}, -\frac{1}{2}, -1, 2, \frac{2}{3}, -\frac{1}{3}, \frac{1}{3}, \ldots$$

This will give an arrangement of $\mathbb{Q}$ into an infinite list with no repeats. So, $\mathbb{Q}$ is countably infinite. $\blacksquare$

sets in each column have same cardinality

Countably infinite

$\mathbb{N}$

$\mathbb{Z}$

$\mathbb{Q}$

Uncountable

$\mathbb{R}$ $\quad \mathcal{P}(\mathbb{R}) \quad \mathcal{P}(\mathcal{P}(\mathbb{R})) \quad \mathcal{P}(\mathcal{P}(\mathcal{P}(\mathbb{R}))) \ldots$

$\mathcal{P}(\mathbb{N})$

$\aleph$ is the first letter in the Hebrew alphabet

aleph

this cardinality is called

$\aleph_0$ aleph naught

We will show $\mathcal{P}(A)$ is "bigger" than A for all A.

is there any set "in between" in cardinality between $\mathbb{N}$ and $\mathbb{R}$?

Unknown.

Continuum hypothesis

You can keep getting "bigger" and "bigger" infinite sets by continually taking the power set of each set.

Some more theorems (see Hammack)

Thm: If A and B are countably infinite, then $A \times B$ is countably infinite. [Similar proof as $\mathbb{Q}$ is countably infinite.]

Thm: If A and B are countably infinite, then $A \cup B$ is countably infinite.

Thm: An infinite subset of a countably infinite set is countably infinite.

Thm: If $U \subseteq A$ and U is uncountable, then A is uncountable.

Thm:

$|\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$

$\boxed{\text{Final}}$

Test 1 topics
Test 2 topics
little bit of cardinality

At least one proof
will be exactly one
from test 1 / test 2
without any changes