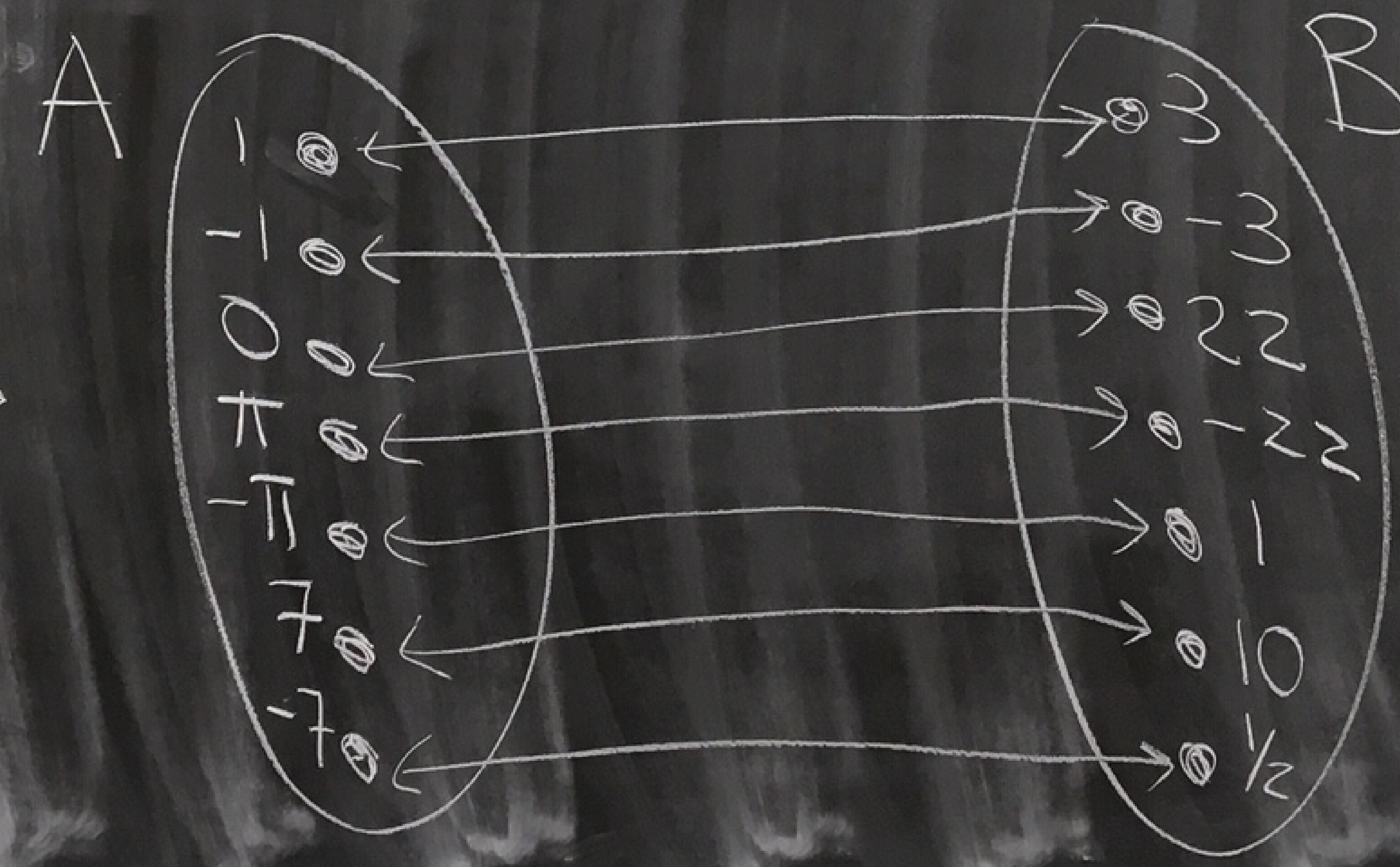
Monday
11/18

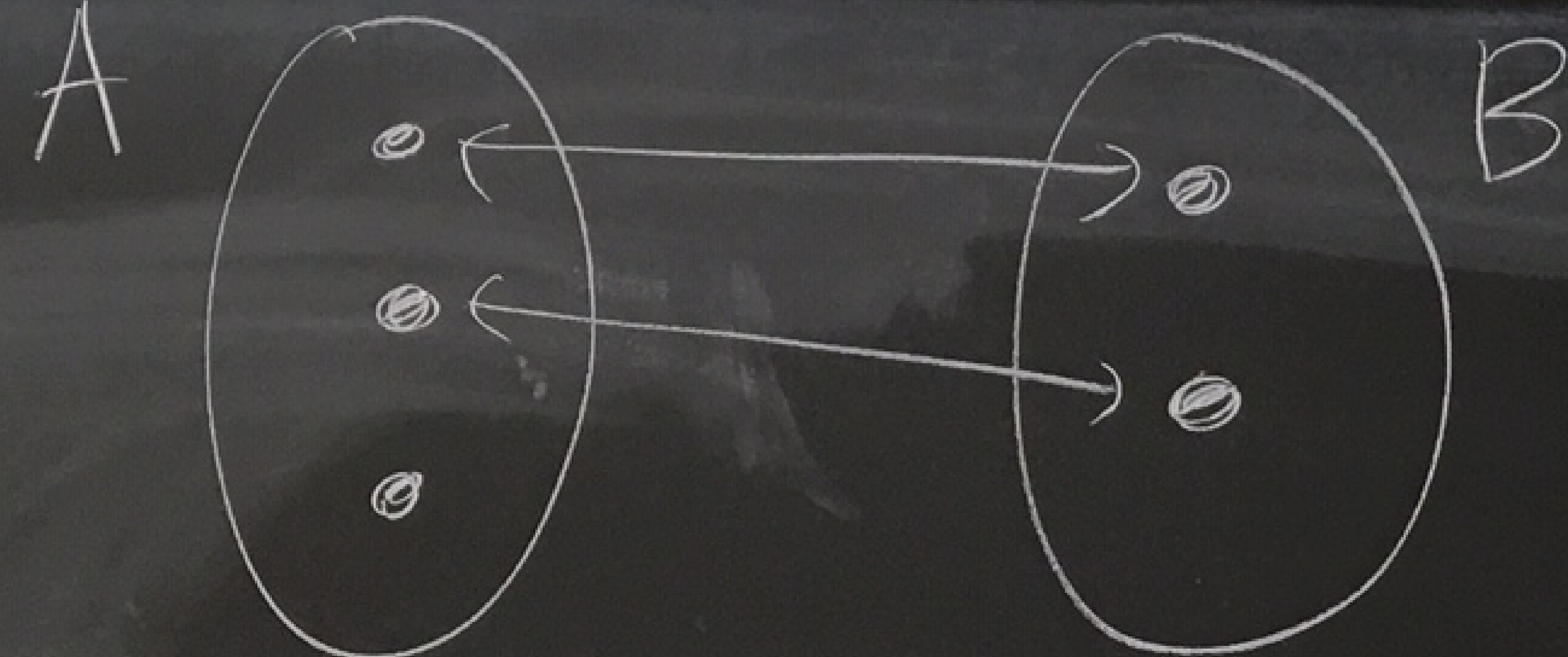
Cardinality

How can we tell if two finite sets have the same number of elements?

Here A and B have the same # of elements



You see if you can pair up the elements in a 1-1 and onto way.



there is no way to pair up the elements of A and B in a 1-1 and onto way.

So, A and B have different sizes.

Def: Let A and B be sets. We say that A and B have the same <u>cardinality</u> if there exists a bijection between them. And if such a bijection exists we write $|A| = |B|$.

If no such bijection exists we say that A and B have <u>unequal cardinality</u> and we write $|A| \neq |B|$.

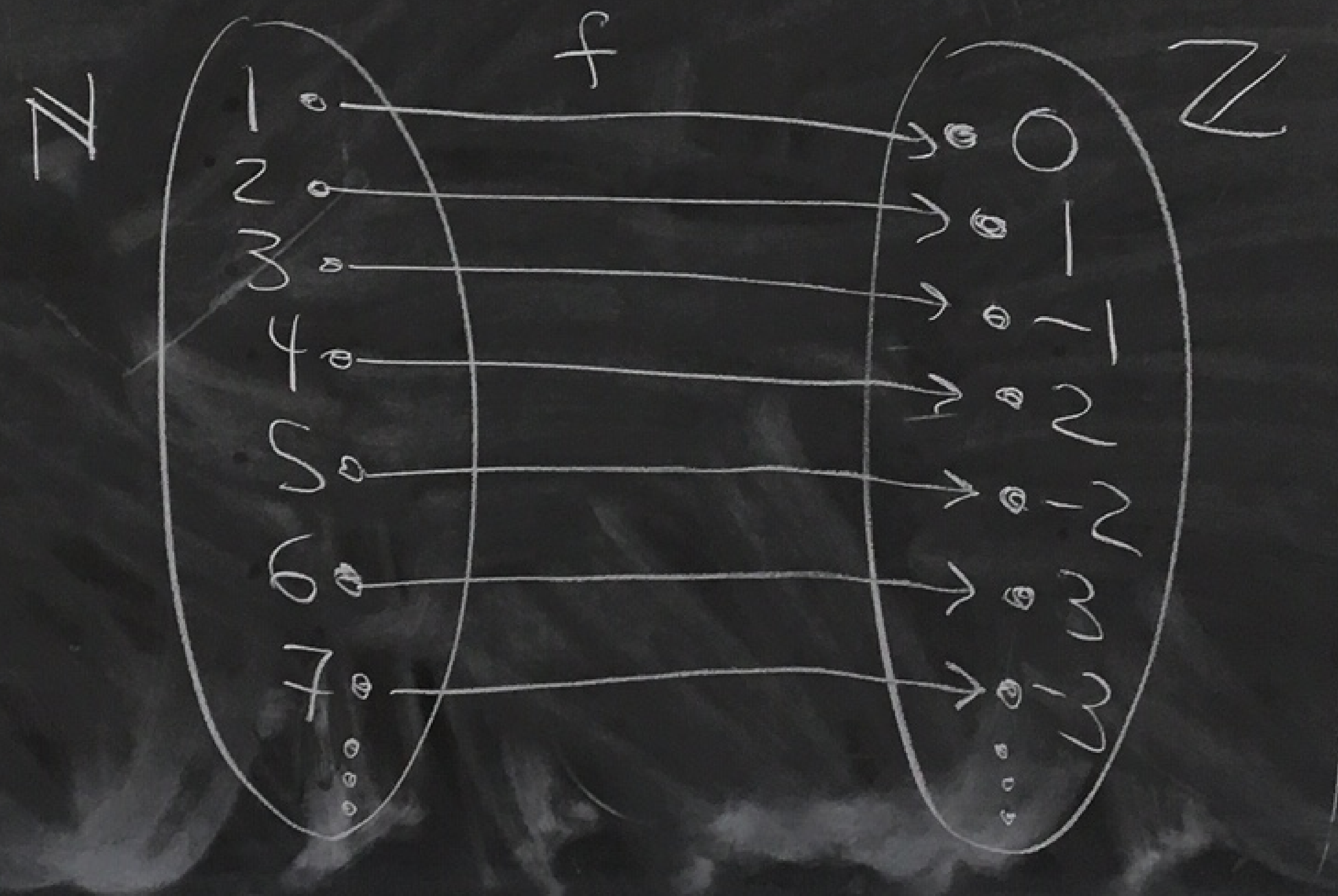
Ex: $\mathbb{N} = \{1, 2, 3, 4, 5, \ldots\}$

$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Q: Is $|\mathbb{N}| = |\mathbb{Z}|$ or $|\mathbb{N}| \neq |\mathbb{Z}|$?

$\boxed{\text{Formula for } f}$

$$f(n) = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ -\left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd} \end{cases}$$



Define $f : \mathbb{N} \to \mathbb{Z}$ as in the picture (and keep going with the same pattern).

Then $f : \mathbb{N} \to \mathbb{Z}$ is a bijec

So, $|\mathbb{N}| = |\mathbb{Z}|$

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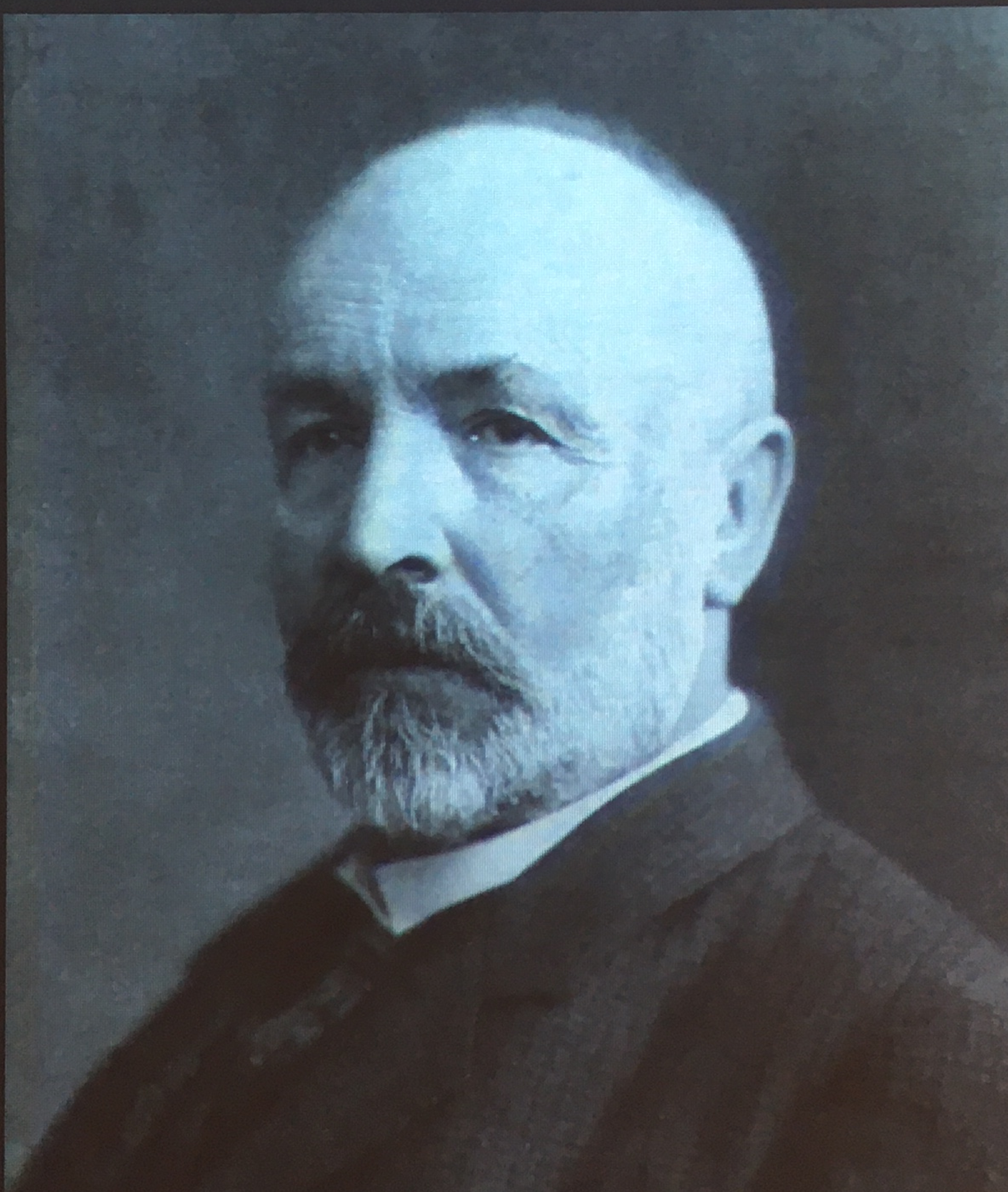

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Ex: $|\mathbb{N}| \neq |\mathbb{R}|$.

(Cantor's Diagonalization Argument)

We will show that there is no bijection $f: \mathbb{N} \to \mathbb{R}$.

Suppose $f: \mathbb{N} \to \mathbb{R}$.
We prove that f can't be onto.

Let's make a table.
Suppose f has the following outputs.

n	$f(n)$
1	$x_1 . b_{11} b_{12} b_{13} b_{14} b_{15} \ldots$
2	$x_2 . b_{21} b_{22} b_{23} b_{24} b_{25} \ldots$
3	$x_3 . b_{31} b_{32} b_{33} b_{34} b_{35} \ldots$
4	$x_4 . b_{41} b_{42} b_{43} b_{44} b_{45} \ldots$
5	$x_5 . b_{51} b_{52} b_{53} b_{54} b_{55} \ldots$
$\vdots$	$\vdots \quad \ddots$

here $x_i \in \mathbb{Z}$

$b_{ij} \in \{0,1,2,3,4,5,6,7,8,9\}$

If the decimal expansion terminates, just add 0's forever.

Define $b = 0.b_1 b_2 b_3 b_4 b_5 \ldots$

where $b_i = \begin{cases} 0 & \text{if } b_{ii} \neq 0 \\ 1 & \text{if } b_{ii} = 0 \end{cases}$

By construction $b \neq f(n)$ for all $n \in \mathbb{N}$.

So, b is not in the range of f.

But $b \in \mathbb{R}$. So, f is not onto. $\blacksquare$

Concrete example

n	$f(n)$
1	$1.32547\ldots$
2	$5.0000000\ldots$
3	$-17.3317325\ldots$
4	$0.5000000\ldots$
5	$3.1415926\ldots$
6	$-100.10057892\ldots$
$\vdots$	$\vdots$

$\boxed{b_{11}=3}$ $\boxed{b_{33}=1}$ $\boxed{b_{55}=9}$

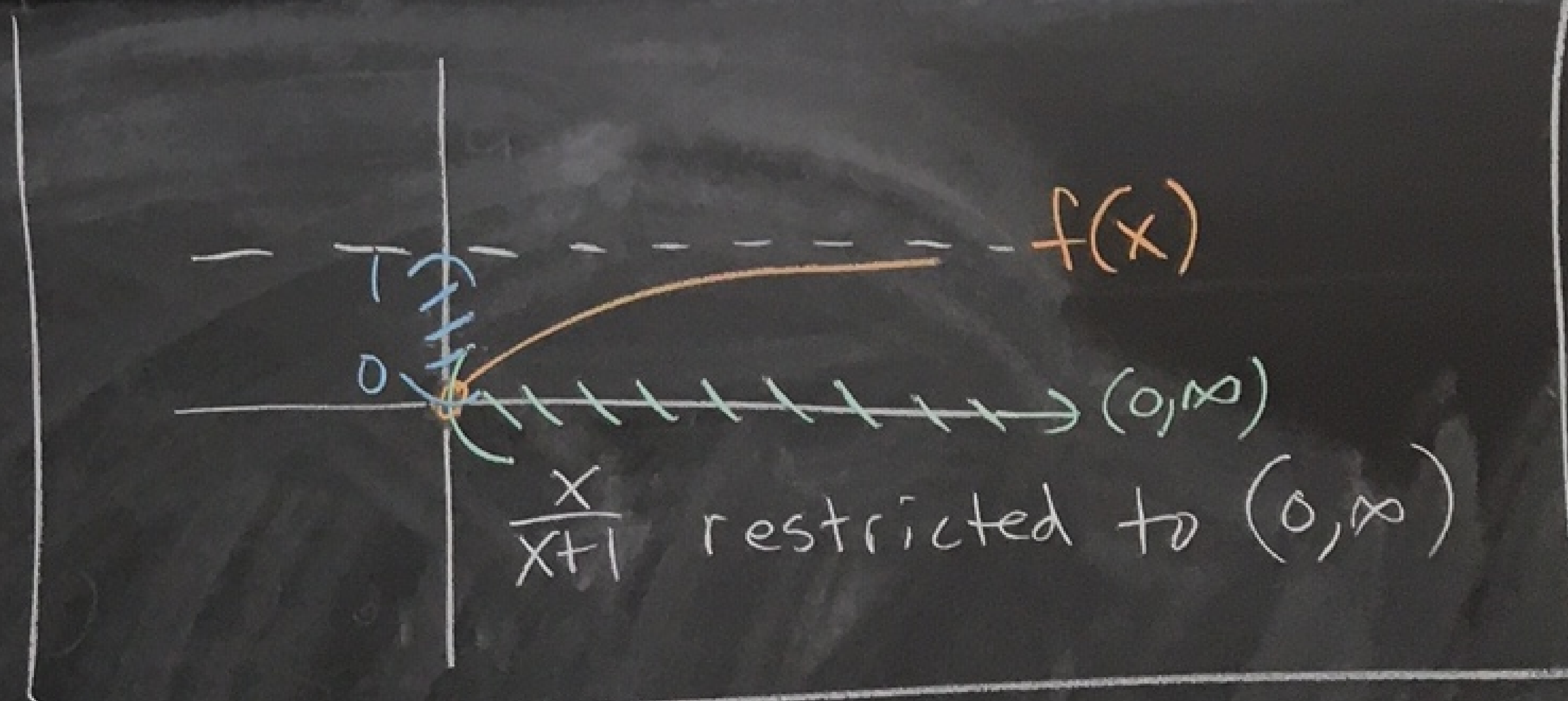
$$b = 0.010100\ldots$$

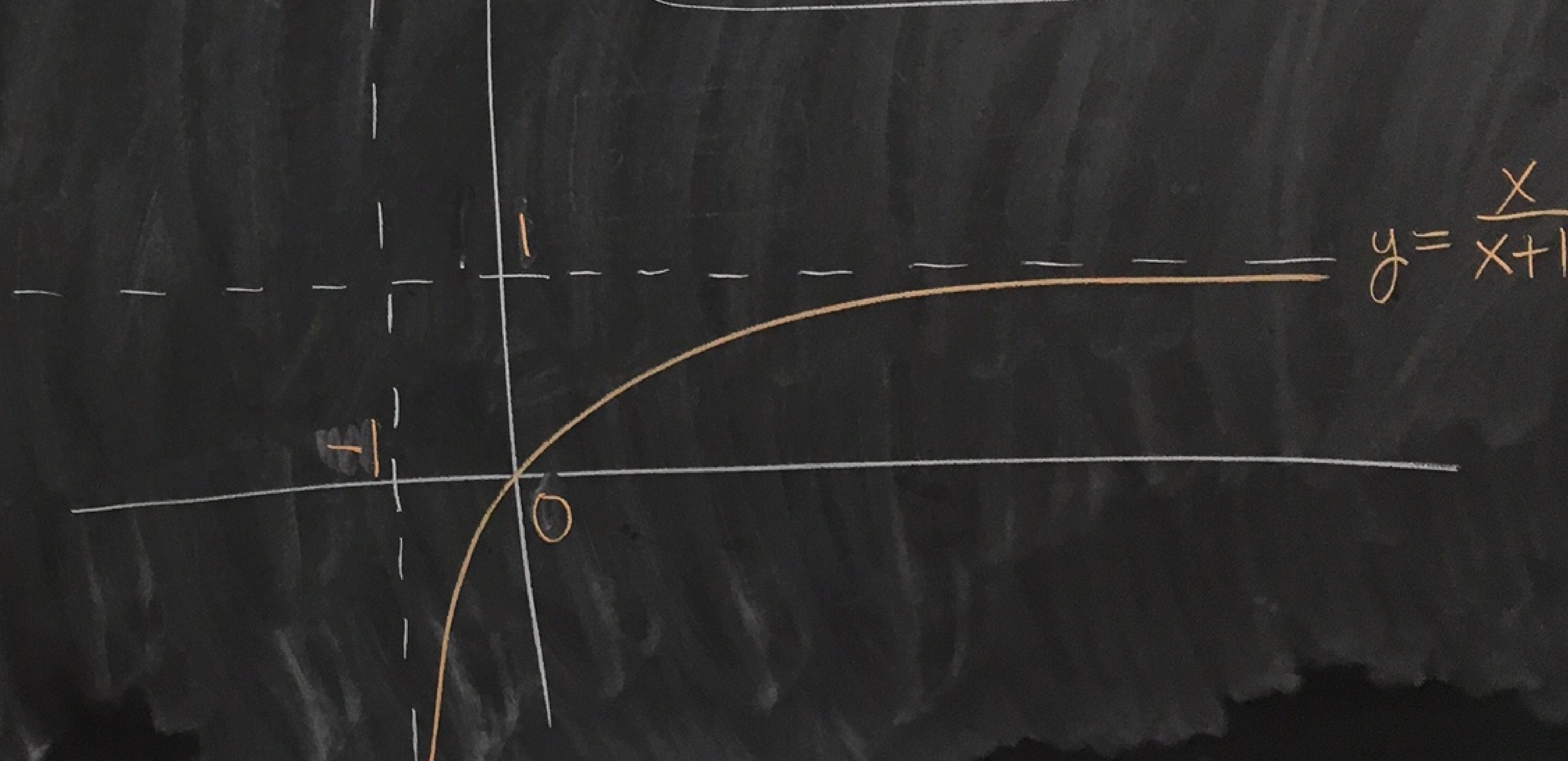
$\boxed{b_{22}=0}$ $\boxed{b_{44}=0}$ $\boxed{b_{66}=8}$

Ex: $|(0,\infty)| = |(0,1)|$

Let $f:(0,\infty) \longrightarrow (0,1)$ be defined by

$$f(x) = \frac{x}{x+1}$$





$\underline{f \text{ is } 1\text{–}1}$

Suppose $f(x_1) = f(x_2)$ where $x_1, x_2 \in (0,\infty)$.

Then $\dfrac{x_1}{x_1+1} = \dfrac{x_2}{x_2+1}$.

So, $x_1x_2 + x_1 = x_1x_2 + x_2$.

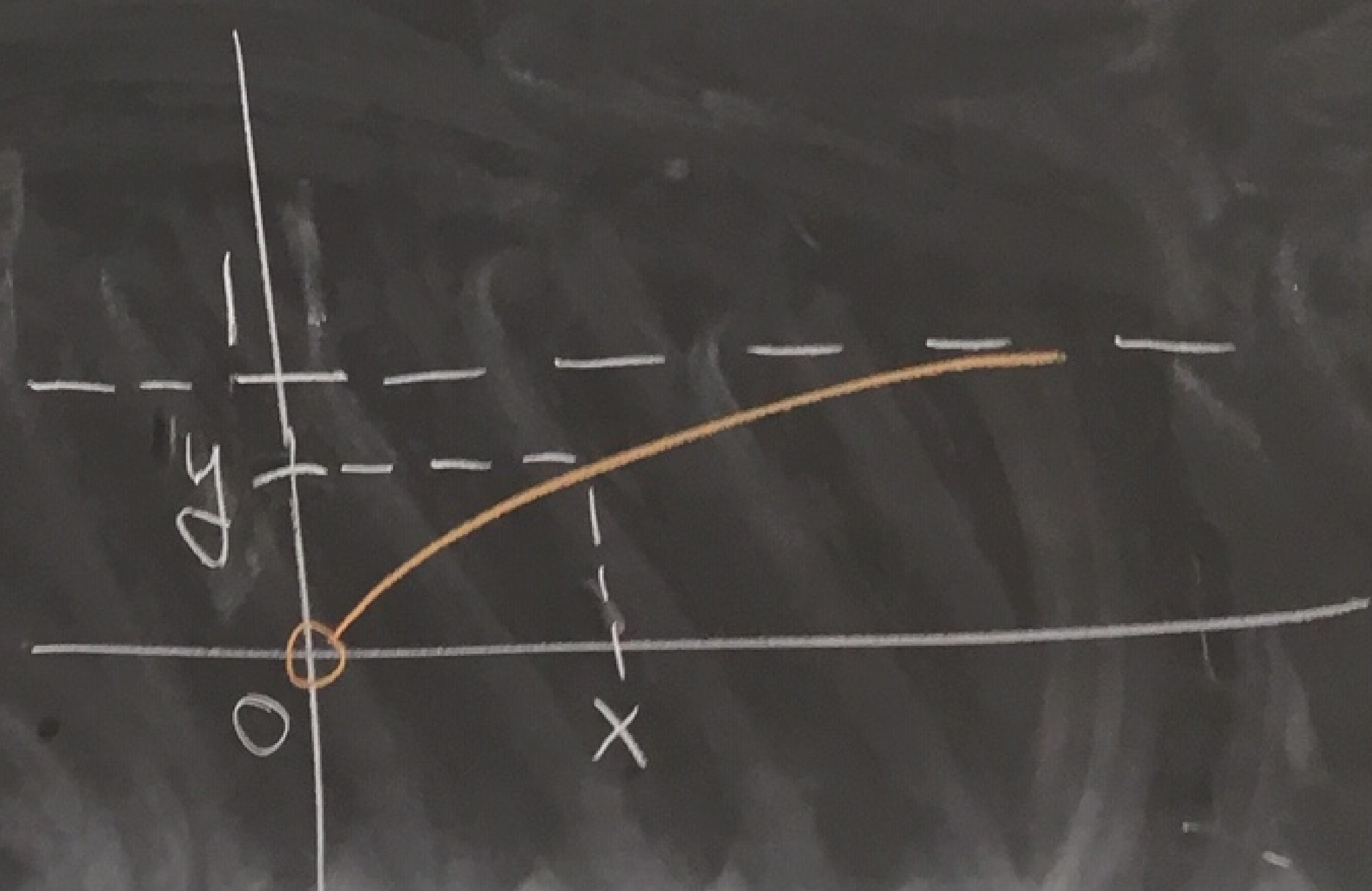
Thus, by adding $-x_1x_2$ to both sides,

we get $x_1 = x_2$.

$\underline{f \text{ is onto}}$

Let $y \in (0,1)$.

We want to find $x \in (0,\infty)$ where $f(x) = y$.



Let's solve $\frac{x}{x+1} = y$.

We want $x = xy + y$,

ie $x - xy = y$,

ie $x(1-y) = y$,

ie $\boxed{x = \frac{y}{1-y}}$

__Check:__ $f\left(\frac{y}{1-y}\right) = \frac{\left(\frac{y}{1-y}\right)}{\left(\frac{y}{1-y}\right)+1}$

$= \frac{\left(\frac{y}{1-y}\right)}{\left(\frac{y+(1-y)}{1-y}\right)} = \frac{y}{1} = y.$

Is $x \in (0,\infty)$?

We know $0 < y < 1$.

So, $-1 < -y < 0$.

Thus, $0 < 1-y < 1$.

Thus, $\boxed{0 < \frac{y}{1-y} = x}$.

So,

$\boxed{|(0,\infty)| = |(0,1)|}$

__Summary:__ Given $y \in (0,1)$,

set $x = \frac{y}{1-y}$. Then $f(x) = y$

and $0 < x < \infty$. So, f is onto. ▨