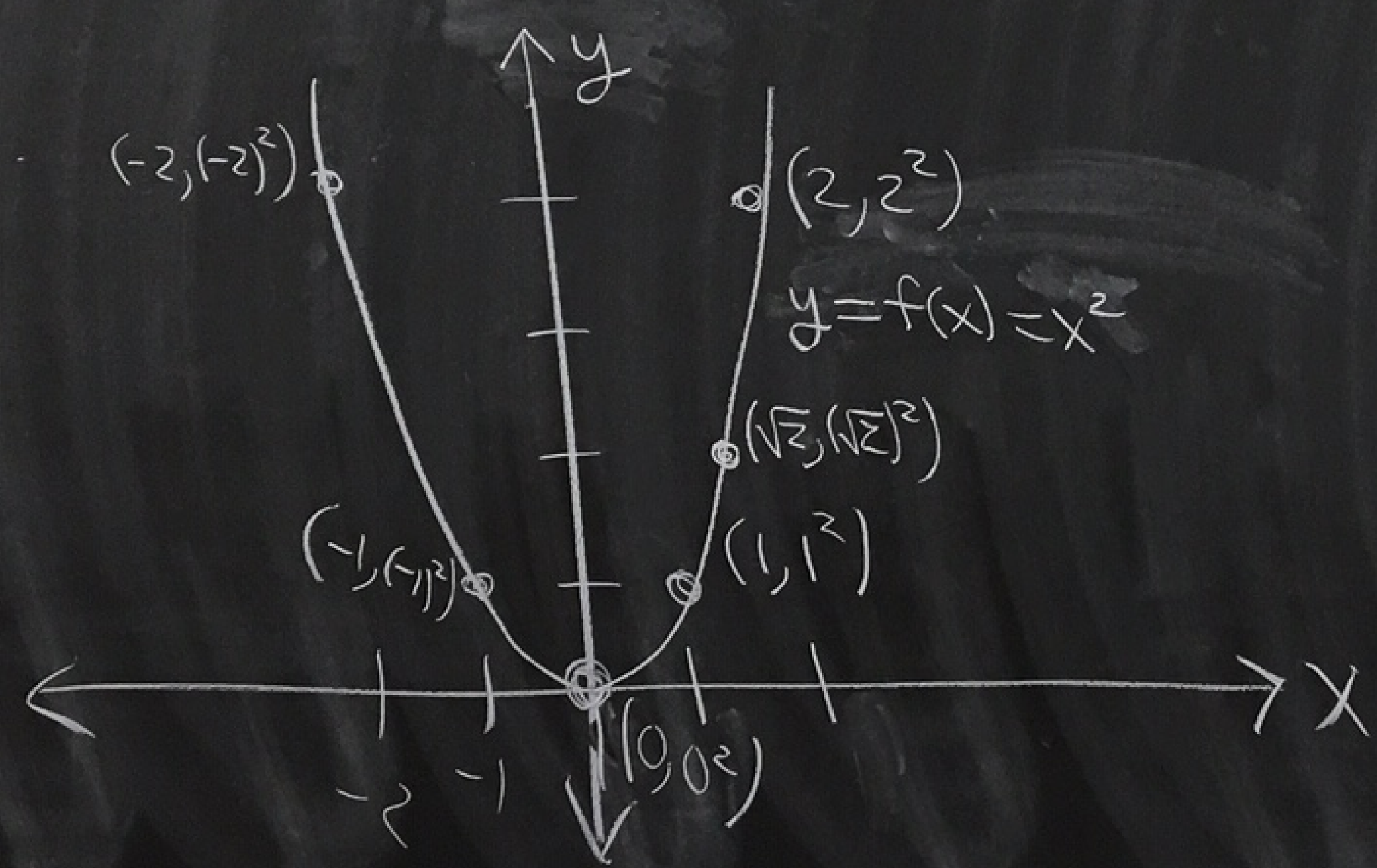
$\boxed{\text{Mon } 10/7}$

Functions (HW 4 material)

Ex: Consider $f(x) = x^2$



How can we think of f as a set?

$$\boxed{\begin{array}{l} f = \{(x, x^2) \mid x \in \mathbb{R}\} \\ \text{Here } f \subseteq \mathbb{R} \times \mathbb{R} \end{array}}$$

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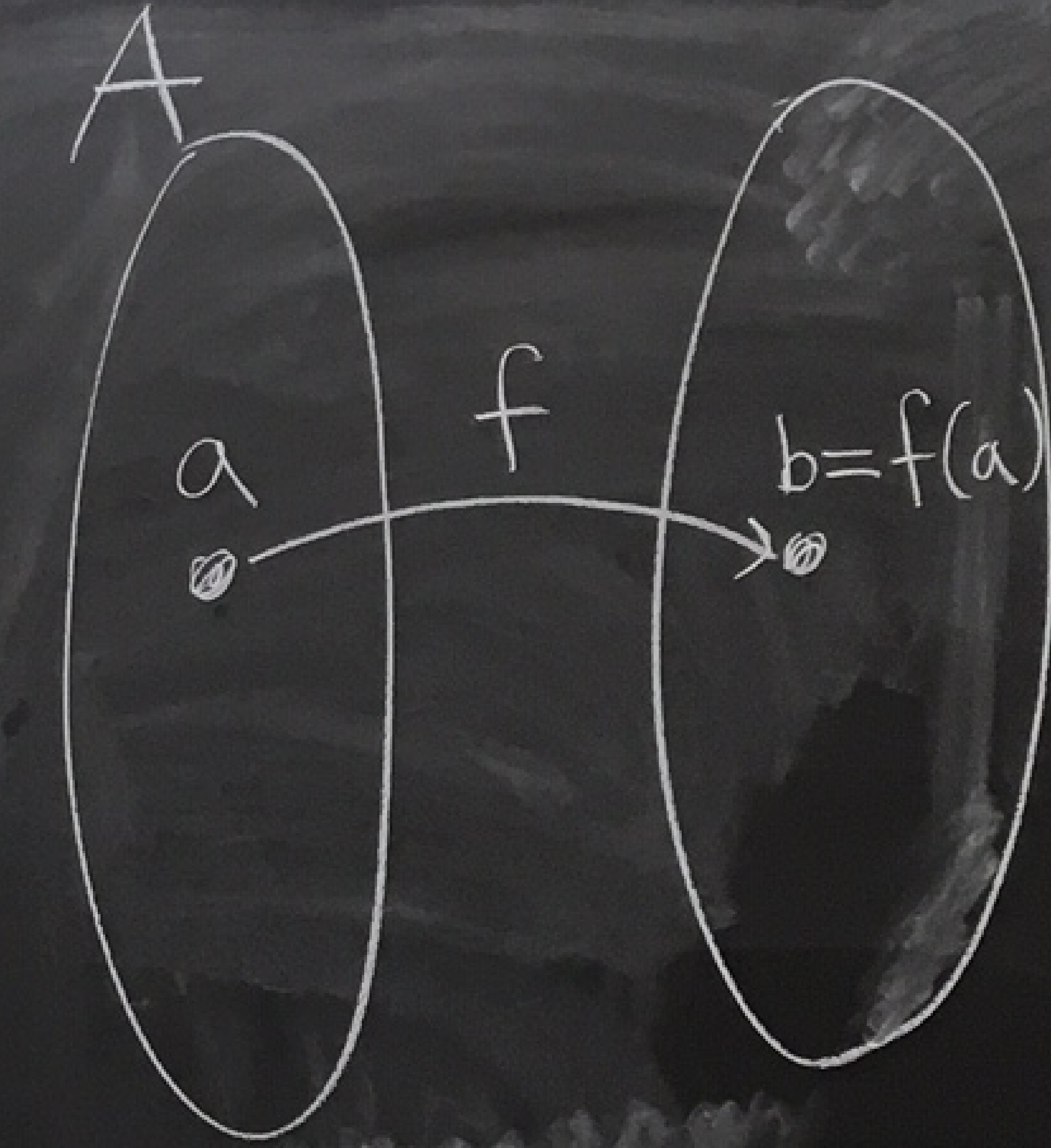
Def: Let A and B be sets.
Let f be a subset of $A \times B$.
We say that f is a function from A to B if

(1) For every $a \in A$ there exists $b \in B$ where $(a,b) \in f$.

This saying that we can plug any $a \in A$ into f and get b. b will be $f(a)$.

and (2) If (a,b_1) and (a,b_2) are in f, then $b_1 = b_2$.

There is a unique b for each a. This is the vertical line test.



If this is the case, then we write

$f: A \to B$ to mean that f is a function from A to B.

- The set A is called the <u>domain</u> of f.
- The set B is called the <u>codomain</u> of f.
- If $(a,b) \in f$ then we write $f(a)=b$.
  If $(a,b) \notin f$ then we write $f(a) \neq b$.
- The <u>range</u> of f is

$$\text{range}(f) = \{ b \in B \mid \exists a \in A \text{ with } f(a) = b \}$$

<u>Recall:</u> $\exists$ means "there exists"

$=b\}$

Ex: $A = \{-1, 100, 3, \frac{72}{10}\}$

$B = \{\pi, -12, -1, \frac{1}{2}, 17, 14\}$

$f = \{(-1,-1), (100, \pi), (3, 17), (\frac{72}{10}, -1)\}$

Is f a function from A to B?

① ✓

② ✓

Yes

$f(-1) = -1$

$f(100) = \pi$

$f(3) = 17$

$f(\frac{72}{10}) = -1$

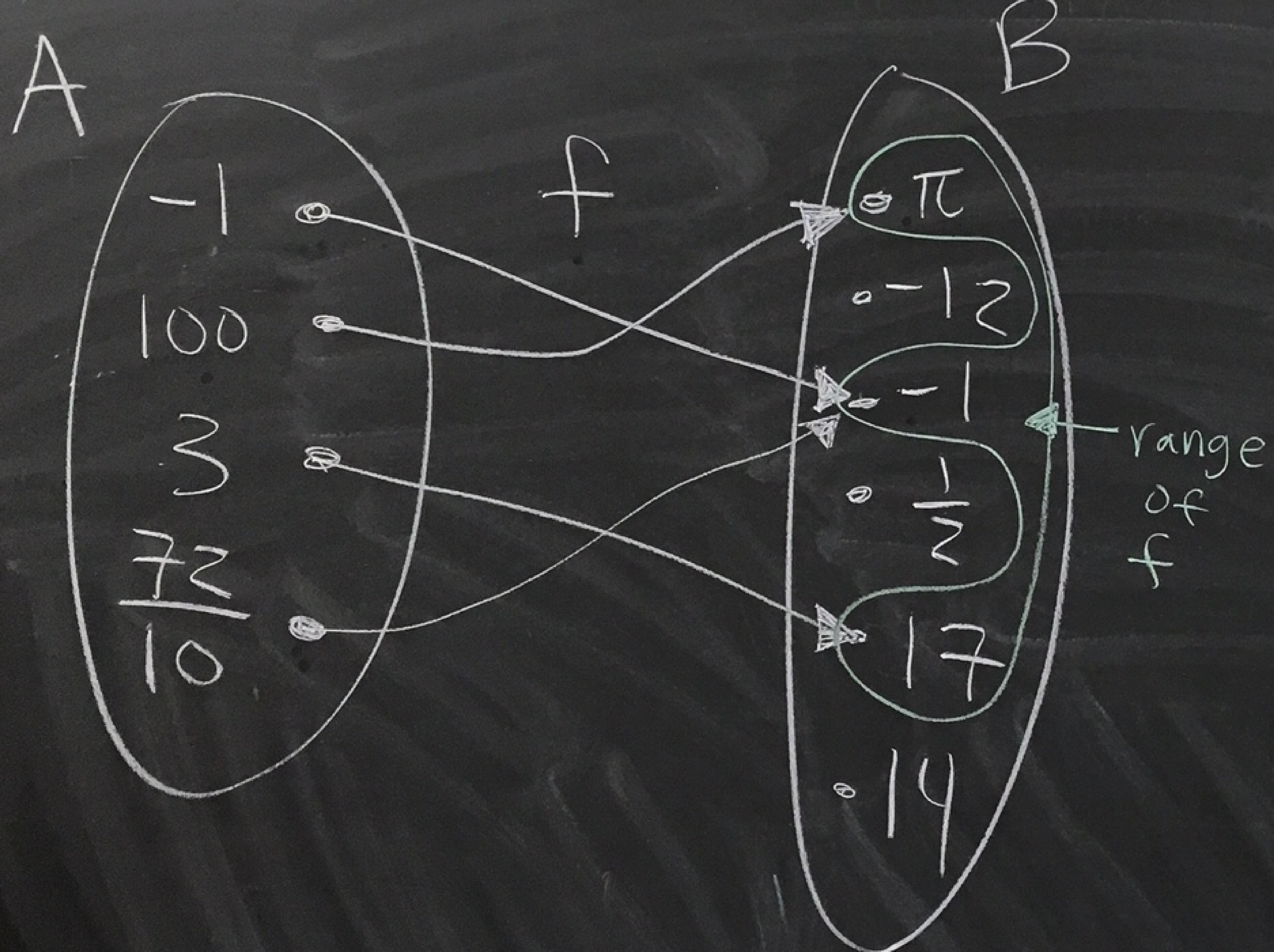
f would not be a function if say $(-1,\pi)$ and $(-1,-12)$ were in f

$(-1, \pi) \leftarrow f(-1) = \pi$

$(-1,-12) \leftarrow f(-1) = -12$

What is $f(-1)$?

Picture of f



$$\text{domain}(f) = A$$

$$\text{codomain}(f) = B$$

$$\text{range}(f) = \{ b \in B \mid \exists\, a \in A \text{ with } f(a) = b \}$$

$$= \{ \pi, -1, 17 \}$$

For example, $\pi \in \text{range}(f)$ since $100 \in A$ and $f(100) = \pi$.
$-12 \notin \text{range}(f)$ since there is no $a \in A$ with $f(a) = -12$.

Ex:

Is

Ex: $A=\{-1, 100, 3, \frac{72}{10}\}$

$B=\{\pi, -12, -1, \frac{1}{2}, 17, 14\}$

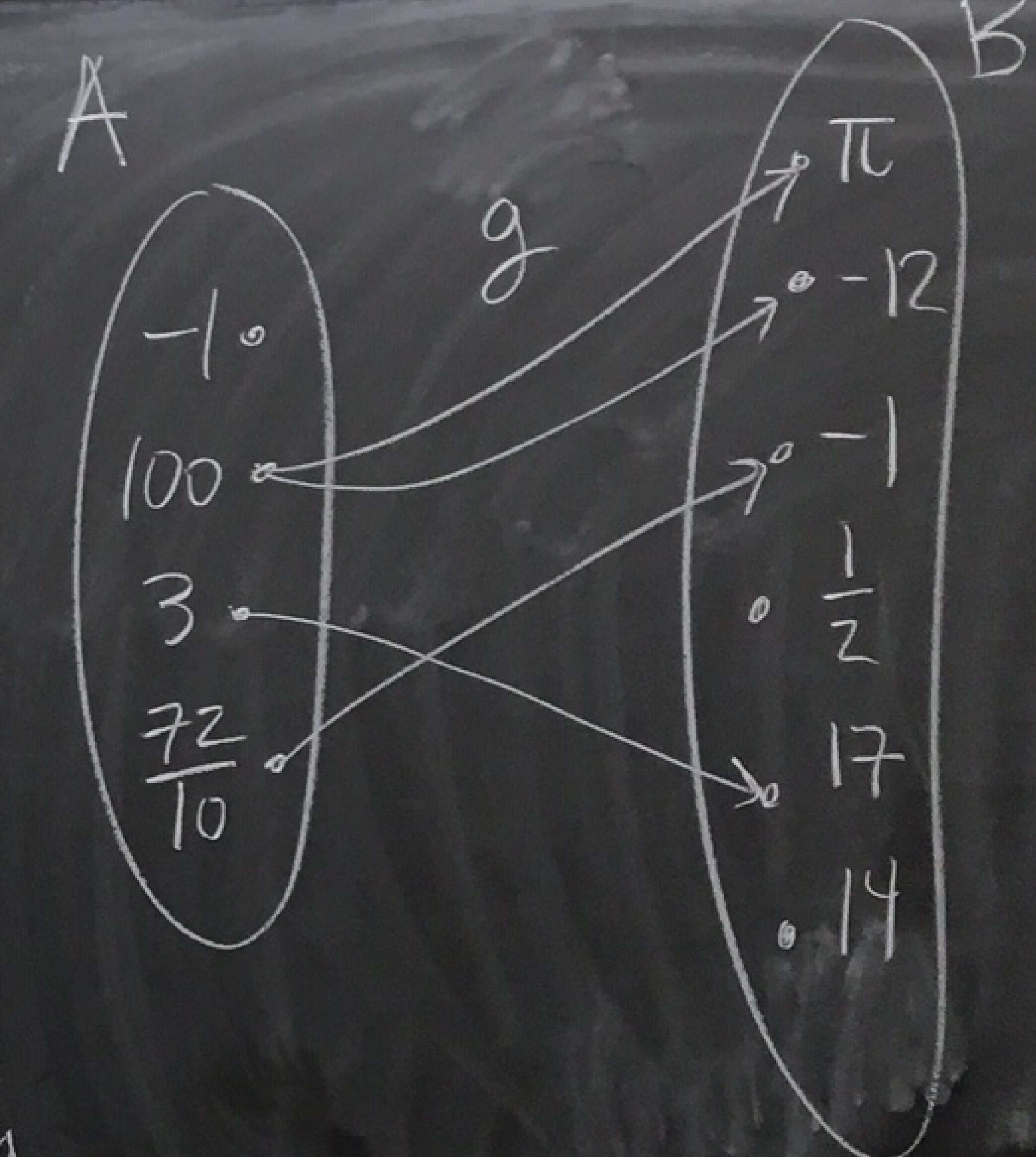
$g=\{(100,\pi), (3,17), (\frac{72}{10},-1), (100,-12)\}$

Is g a function from A to B? NO

① There is no $b\in B$ where $(-1,b)\in g$ or $g(-1)=b$.

① is not satisfied.

② $(100,\pi)$ and $(100,-12)$ are both in g. Thats saying $g(100)=\pi$ and $g(100)=-12$. That's no good.



Test review — TOPICS

- HW 1 - set builder notation
- HW 2 — set theory
- HW 3 — equivalence relations
  modulo n.

Structure

- calculations (4 prob.)
- proofs (3 prob.)

Hw 2

② Let $A = \{2k \mid k \in \mathbb{Z}\}$
and $B = \{3n \mid n \in \mathbb{Z}\}$
Prove $A \cap B = \{6m \mid m \in \mathbb{Z}\}$

proof:

$\boxed{\subseteq}$: Let $x \in A \cap B$.
Then $x \in A$ and $x \in B$.

So, $x = 2k$ where $k \in \mathbb{Z}$
and $x = 3n$ where $n \in \mathbb{Z}$.

Thus, $2k = 3n$.

Note that n cannot be odd because if n was odd then $3n$ would be odd. But $3n = 2k$ is even.

So n is even.
Thus $n = 2l$ where $l \in \mathbb{Z}$.
So, $x = 3n = 3(2l) = 6l \in \{6m \mid m \in \mathbb{Z}\}$.
So, $A \cap B \subseteq \{6m \mid m \in \mathbb{Z}\}$

$\boxed{\supseteq}$: Let $y \in \{6m \mid m \in \mathbb{Z}\}$.

Then, $y = 6m$ where $m \in \mathbb{Z}$.

So, $y = 6m = 2(3m) \in A$

And, $y = 6m = 3(2m) \in B$.

Thus, $y \in A \cap B$.

Therefore, $\{6m \mid m \in \mathbb{Z}\}$.

By $\boxed{\subseteq}$ and $\boxed{\supseteq}$ we have $A \cap B = \{6m \mid m \in \mathbb{Z}\}$ ▨

Hi

2

pro

$\boxed{\subseteq}$

Hw 2

(15) Let A and B be sets.
Prove that $A-B$ and B are disjoint.

pf: We need to show that $(A-B)\cap B=\phi$.

Suppose that $(A-B)\cap B\neq\phi$.

Then there exists $x\in(A-B)\cap B$.

So, $x\in A-B$ and $x\in B$.

Thus, $x\in A$ and $x\notin B$, and $x\in B$.

We can't have $x\notin B$ and $x\in B$.

This is ridiculous!

So, $(A-B)\cap B=\phi$.

Thus, $A-B$ and B are disjoint. $\blacksquare$

Hammock

Chapter 8

13 Let A, B, C be sets. Then

$$A-(B\cup C)=(A-B)\cap(A-C).$$

Proof:

$\boxed{\subseteq}$: Let $x \in A-(B\cup C)$.

Then $x \in A$ and $\boxed{x \notin B\cup C}$.

What does $x \notin B \cup C$ mean?

It means that "$x \in B \cup C$" is not true.

We need the negation of "$x \in B$ or $x \in C$".

So "$x \notin B$ and $x \notin C$" is true.

$\neg(P \text{ or } Q)$

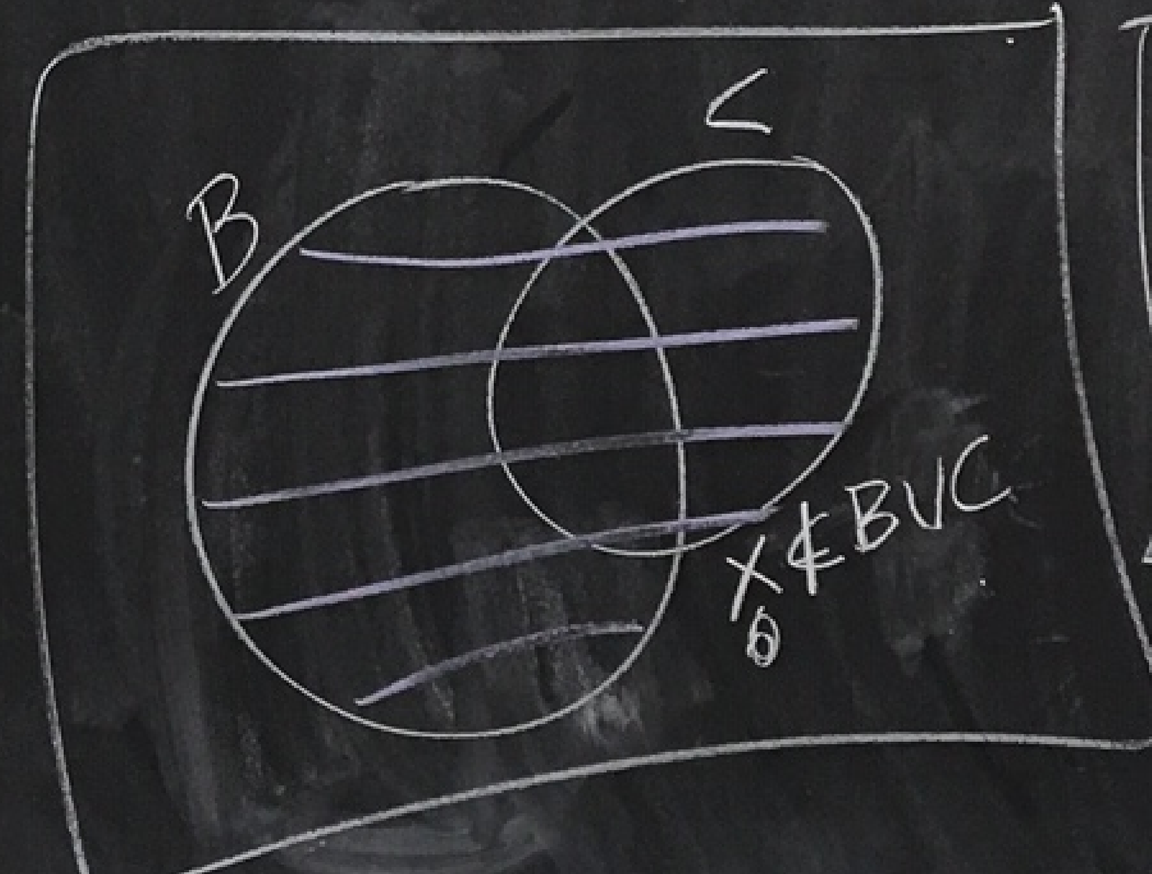
$(\neg P)$ and $(\neg Q)$

In summary, we have

$$x \in A \text{ and } \underbrace{x \notin B \text{ and } x \notin C}_{\text{from } x \notin B \cup C},$$

So, $x \in (A-B)$ and $x \in (A-C)$.

Thus, $x \in (A-B) \cap (A-C)$.



$\equiv$ is $B \cup C$

$\boxed{\supseteq}$: Let $x \in (A-B) \cap (A-C)$.

Then $x \in A-B$ and $x \in A-C$.

So, $x \in A$ and $x \notin B$, and $x \in A$ and $x \notin C$.

Thus, $x \in A$ and $\boxed{x \notin B \text{ and } x \notin C}$.

So, $x \in A$ and $\boxed{x \notin B \cup C}$.

Thus, $x \in A-(B \cup C)$.

Therefore, $(A-B) \cap (A-C) \subseteq A-(B \cup C)$. $\blacksquare$

Hammock 11.3

7 Define $\sim$ on $\mathbb{Z}$ by

$a \sim b$ iff $3a - 5b$ is even.

Prove $\sim$ is an equivalence relation

pf:

(reflexive) Let $x \in \mathbb{Z}$.

Then $3x - 5x = -2x = 2(-x)$ is even.

So, $x \sim x$.

(symmetric) Let $x, y \in \mathbb{Z}$.

Suppose $x \sim y$.

Then, $3x - 5y$ is even.

So, $3x - 5y = 2\Delta$ where $\Delta \in \mathbb{Z}$.

Adding $-8x + 8y$ to both sides yields

$$3y - 5x = \underbrace{2(\Delta - 4x + 4y)}_{\text{even}}.$$

So, $y \sim x$.

Scratchwork

Given: $3x - 5y = 2\Delta \leftarrow x \sim y$

Want: $3y - 5x = 2(\text{integer}) \leftarrow y \sim x$

$$3x - 5y = 2\Delta$$
$$-8x + 8y = 2\Delta - 8x + 8y$$
$$-5x + 3y = 2(\Delta - 4x + 4y)$$

Scratchwork

Given: $3x-5y=2k$

$3y-5z=2\ell$

Want: $3x-5z=2(?)$

Add

(transitive). Let $x,y,z \in \mathbb{Z}$.

Suppose $x \sim y$ and $y \sim z$.

Then, $3x-5y=2k$

and $3y-5z=2\ell$ where $k,\ell \in \mathbb{Z}$.

Adding these equations produces

$$3x-2y-5z=2k+2\ell.$$

So,

$$3x-5z=2(k+\ell+y).$$

Thus, $x \sim z$. $\blacksquare$

MATH 4460 HW #4

(13) Prove that

$$15x^2 - 7y^2 = 1$$

has no integer solutions.

pf: (By contradiction)
Suppose there exist $x, y \in \mathbb{Z}$ with $15x^2 - 7y^2 = 1$.

So in $\mathbb{Z}_7 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}\}$ we have

$$\overline{15x^2 - 7y^2} = \bar{1}.$$

Then,

$$\overline{15}\,\bar{x}^2 + \overline{-7}\,\bar{y}^2 = \bar{1}$$

in $\mathbb{Z}_7$.

In $\mathbb{Z}_7$,
$\bar{7} = \bar{0}$
$\overline{15} = \bar{1}$

So, $\bar{x}^2 = \bar{1}$ in $\mathbb{Z}_7$

But $\bar{x} = \bar{1}$ works in $\mathbb{Z}_7$.
So this leads nowhere.

Let's look in $\mathbb{Z}_3$ now.

So in $\mathbb{Z}_3 = \{\bar{0}, \bar{1}, \bar{2}\}$ we have

$$\overline{15}\,\bar{x}^2 + \overline{-7}\,\bar{y}^2 = \bar{1}.$$

In $\mathbb{Z}_3$

$\overline{15} = \bar{0}$

$\overline{-7} = \bar{2}$

So, $\bar{2}\,\bar{y}^2 = \bar{1}$ in $\mathbb{Z}_3$.

There is no y with $\bar{2}\bar{y}^2 = \bar{1}$ in $\mathbb{Z}_3$ by the following table.

In $\mathbb{Z}_3$

$\bar{y}$	$\bar{2}\bar{y}^2$
$\bar{0}$	$\bar{2}\cdot\bar{0}^2 = \bar{0}$
$\bar{1}$	$\bar{2}\cdot\bar{1}^2 = \bar{2}$
$\bar{2}$	$\bar{2}\cdot\bar{2}^2 = \bar{8} = \bar{2}$

$\bar{2}\bar{y}^2 = \bar{1}$ has no solutions in $\mathbb{Z}_3$

Hence, contradiction. So, there are no $x, y \in \mathbb{Z}$ with $15x^2 - 7y^2 = 1$. ∎

MATH 44

(13) Prove