Weds 10/30

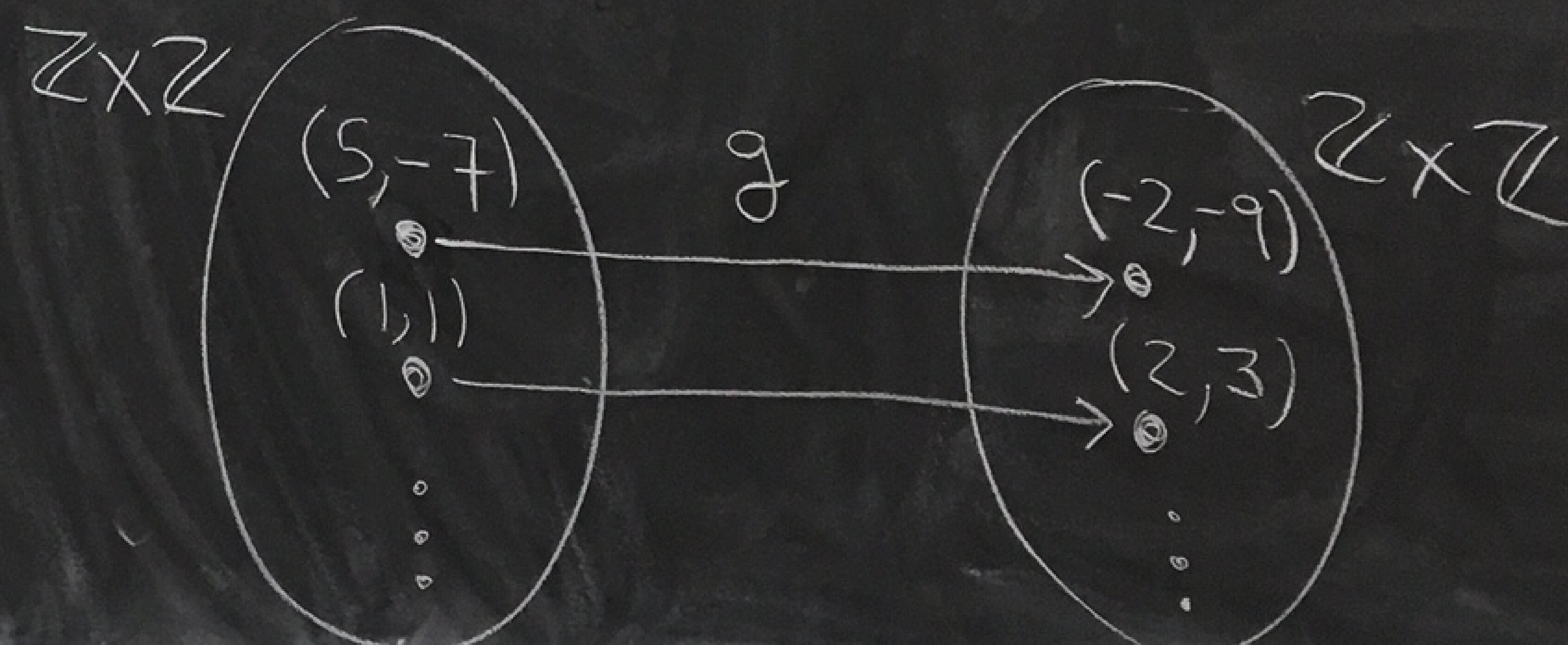
$$g(5,-7) = (5-7,\ 5+2(-7))$$
$$= (-2,-9)$$

Last time

$$g : \mathbb{Z}\times\mathbb{Z} \longrightarrow \mathbb{Z}\times\mathbb{Z}$$

$$g(m,n) = (m+n,\ m+2n)$$

We showed that g is one-to-one.



<u>Claim:</u> g is onto.

<u>proof:</u>

Pick some $(b,c) \in \mathbb{Z}\times\mathbb{Z}$.

We need to find

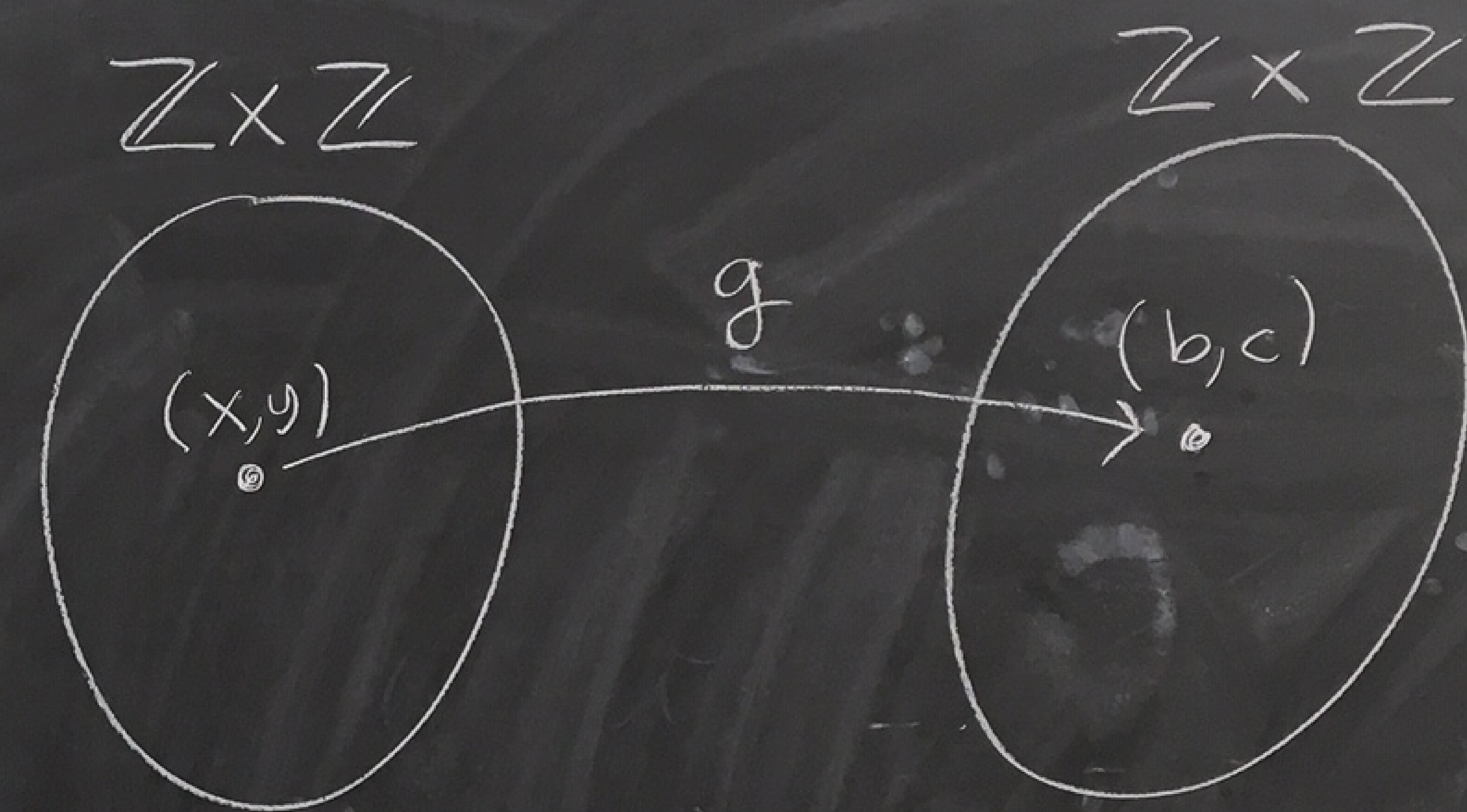
$(x,y) \in \mathbb{Z}\times\mathbb{Z}$ where

$g(x,y) = (b,c)$.

That is we need to solve

$$(x+y, x+2y) = (b,c)$$

for x and y.



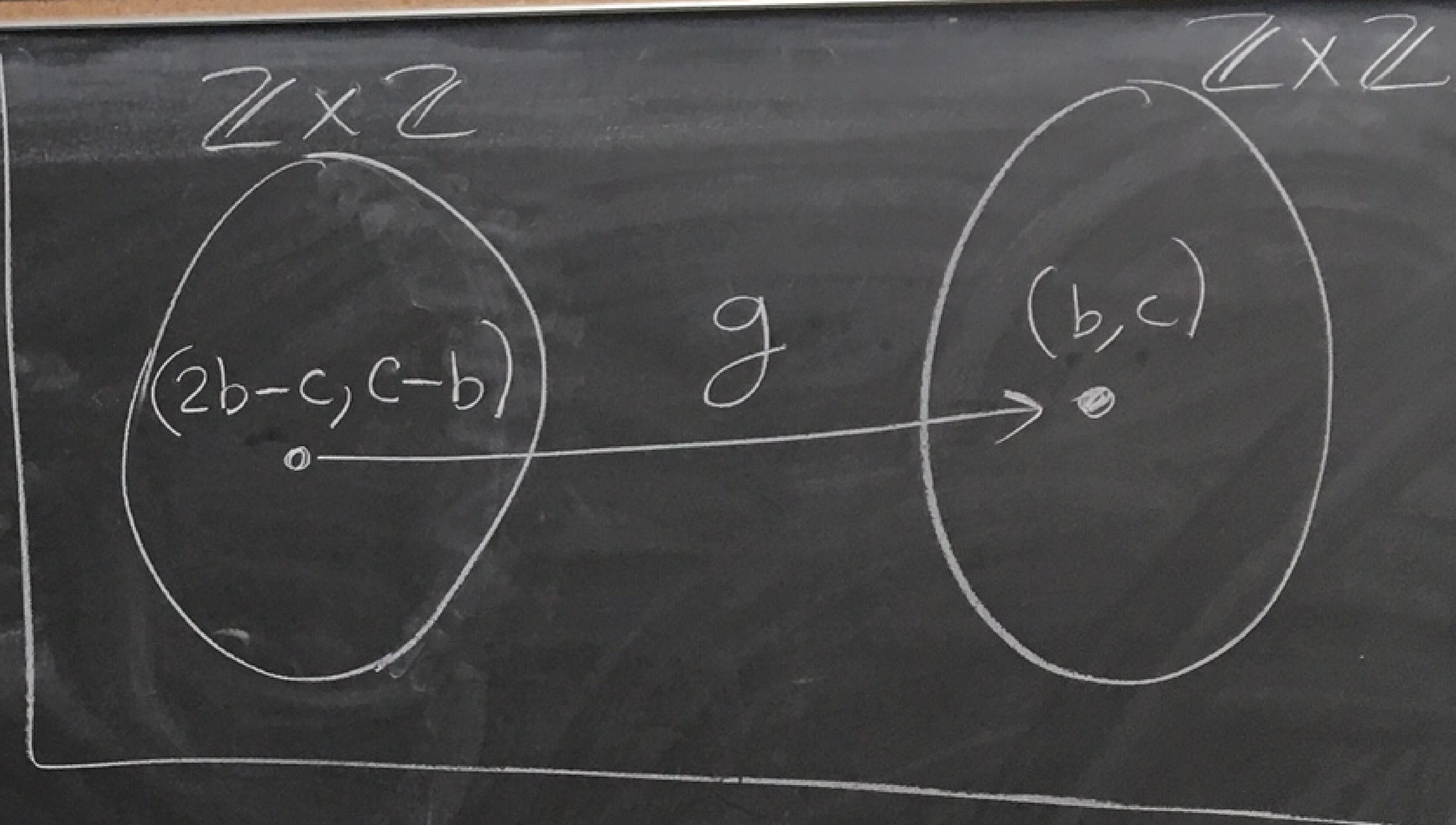
So we need to solve

$$\boxed{\begin{aligned} x+y &= b \quad (1) \\ x+2y &= c \quad (2) \end{aligned}}$$

$-① + ②$ gives $y = c - b$.
Plugging this back into ①
gives $x = b - y = b - (c - b)$
$= 2b - c$.

Note that $(x,y) = (2b-c, c-b)$
is in $\mathbb{Z}\times\mathbb{Z}$ and

$$g(x,y) = g(2b-c, c-b)$$
$$= ((2b-c)+(c-b), (2b-c)+2(c-b))$$
$$= (b,c).$$



Summary: Given $(b,c) \in \mathbb{Z}\times\mathbb{Z}$ we have that $(2b-c, c-b) \in \mathbb{Z}\times\mathbb{Z}$
and $g(2b-c, c-b) = (b,c)$. So, g is onto.

So, $g: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$

given by $g(m,n) = (m+n, m+2n)$

is a bijection (one-to-one & onto).

Since g is one-to-one, g^{-1} exists.

And $\text{domain}(g^{-1}) = \text{range}(g) = \mathbb{Z}\times\mathbb{Z}$.

g is onto

So, $g^{-1}: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$.

Claim: $g^{-1}(b,c) = (2b-c, c-b)$

pf: Let $h: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ with $h(b,c) = (2b-c, c-b)$.

Thm from last time (part 6)

$f: A \to B$, f is 1-1

$C = \text{range}(f)$

If $h: C \to A$ and $h\circ f = i_A$ then $h = f^{-1}$.

Let's show $h\circ g = i$.

Let $(m,n) \in \mathbb{Z}\times\mathbb{Z}$.

Then,

$$(h\circ g)(m,n) = h(g(m,n)) = h(m+n, m+2n)$$

$$= (2(m+n)-(m+2n), (m+2n)-(m+n))$$

$$= (m,n) = i(m,n).$$

By thm part 6 from last time $g^{-1} = h$.

$\boxtimes$

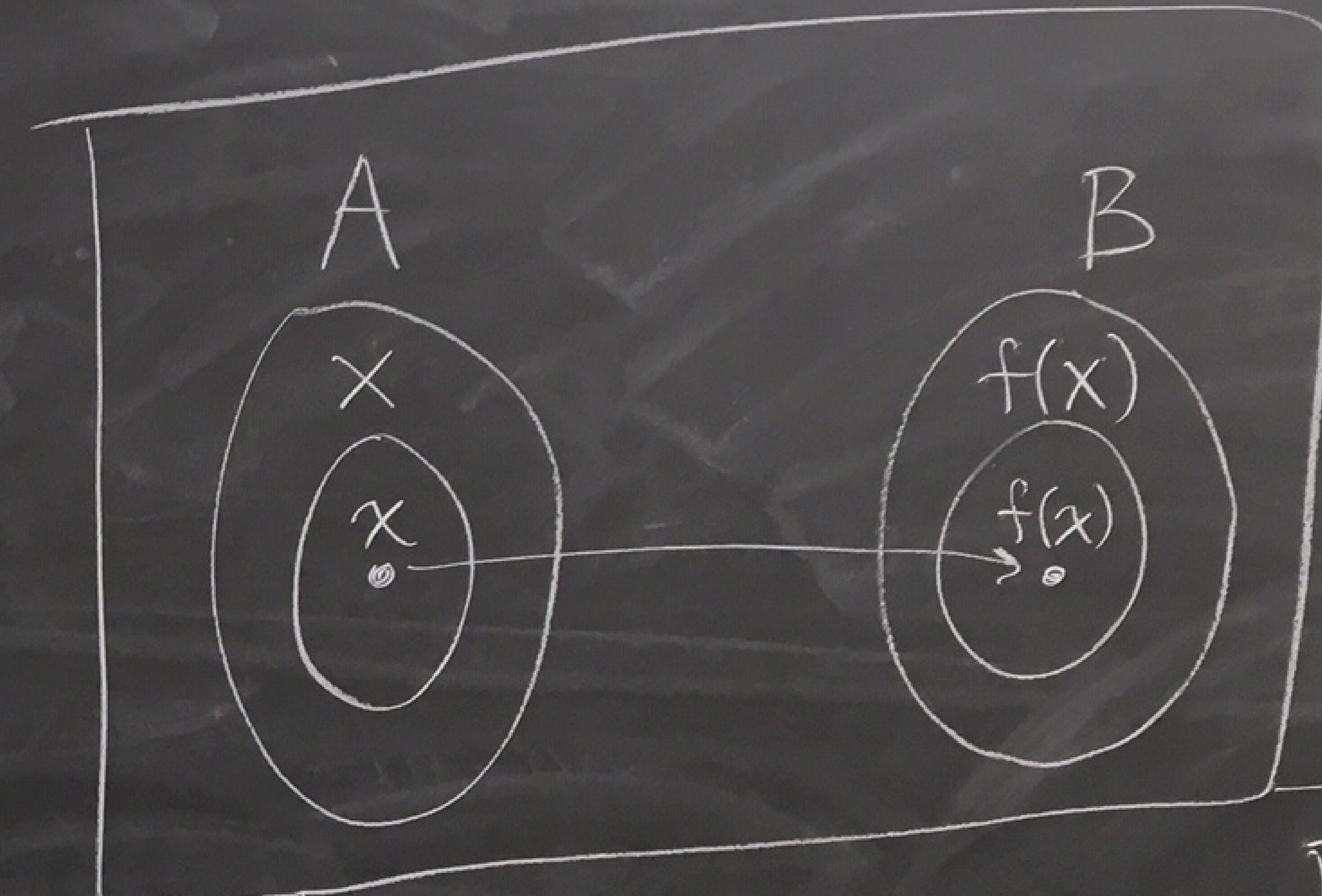
<u>Def</u>: Let A and B be sets.

Let $f: A \to B$.

① Let $X \subseteq A$.

The <u>image of X under f</u> is

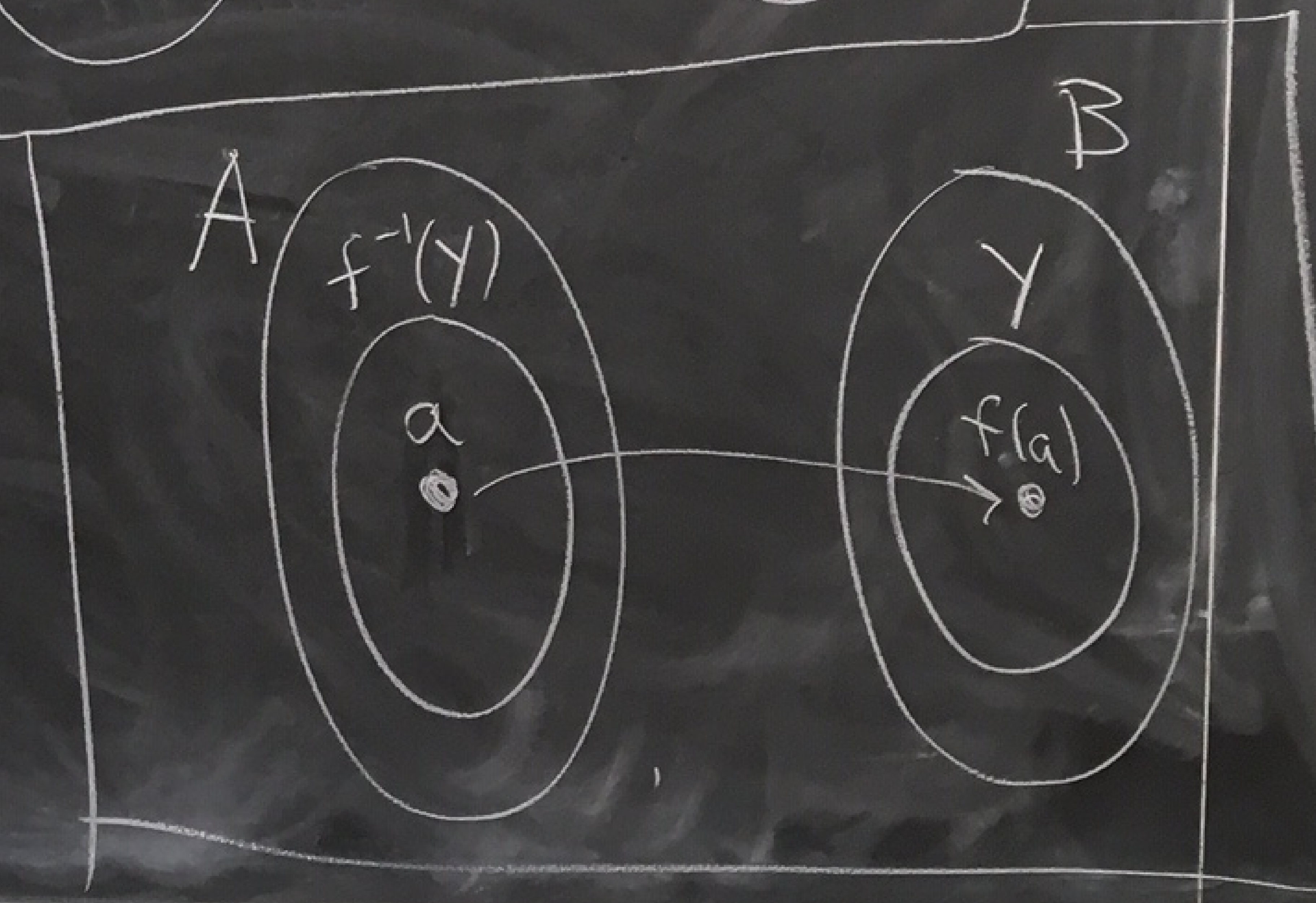
$$f(X) = \{ f(x) \mid x \in X \}$$



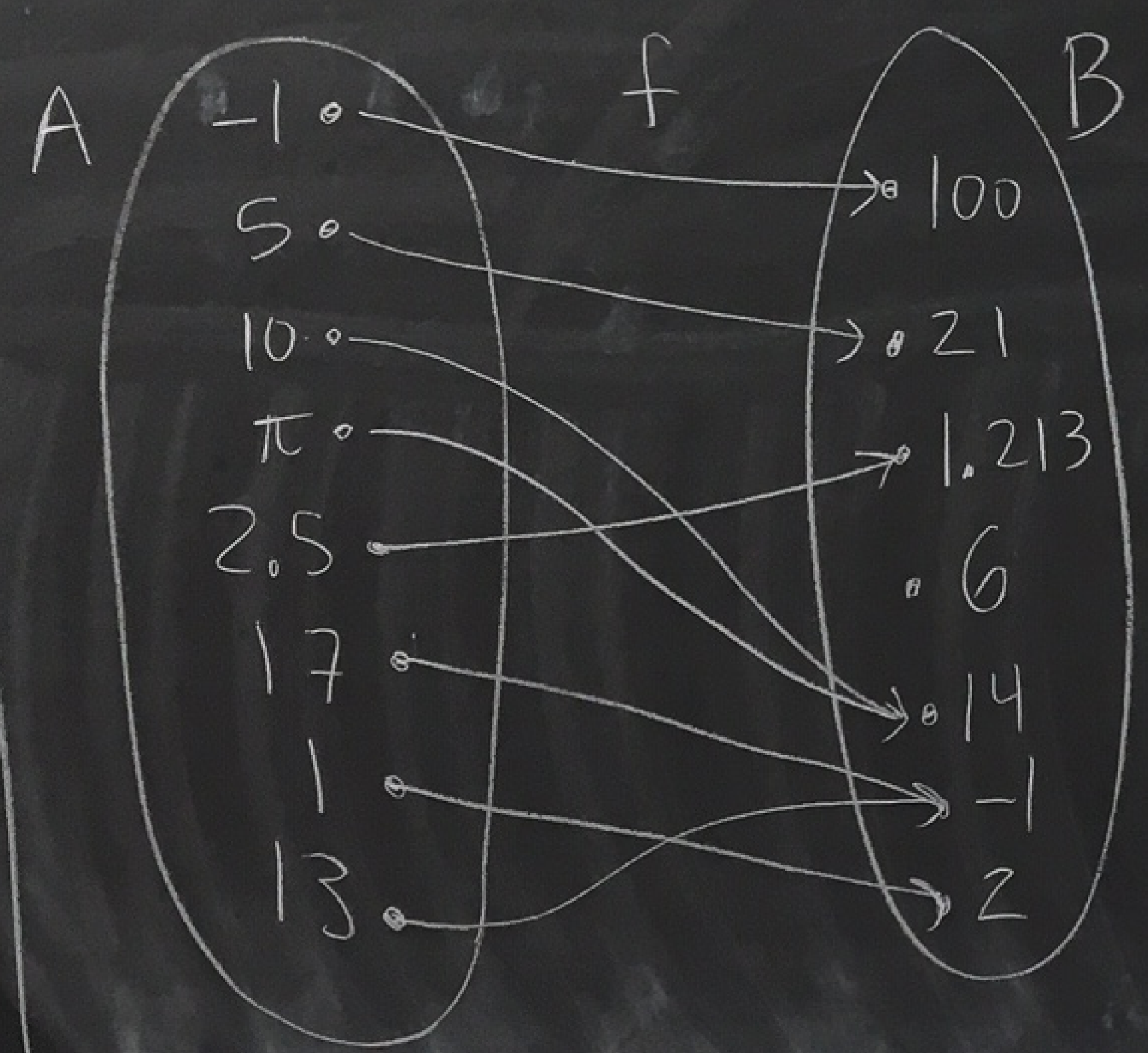
② Let $Y \subseteq B$.

The <u>inverse image of Y under f</u> is

$$f^{-1}(Y) = \{ a \in A \mid f(a) \in Y \}$$



Ex: Consider the following function.

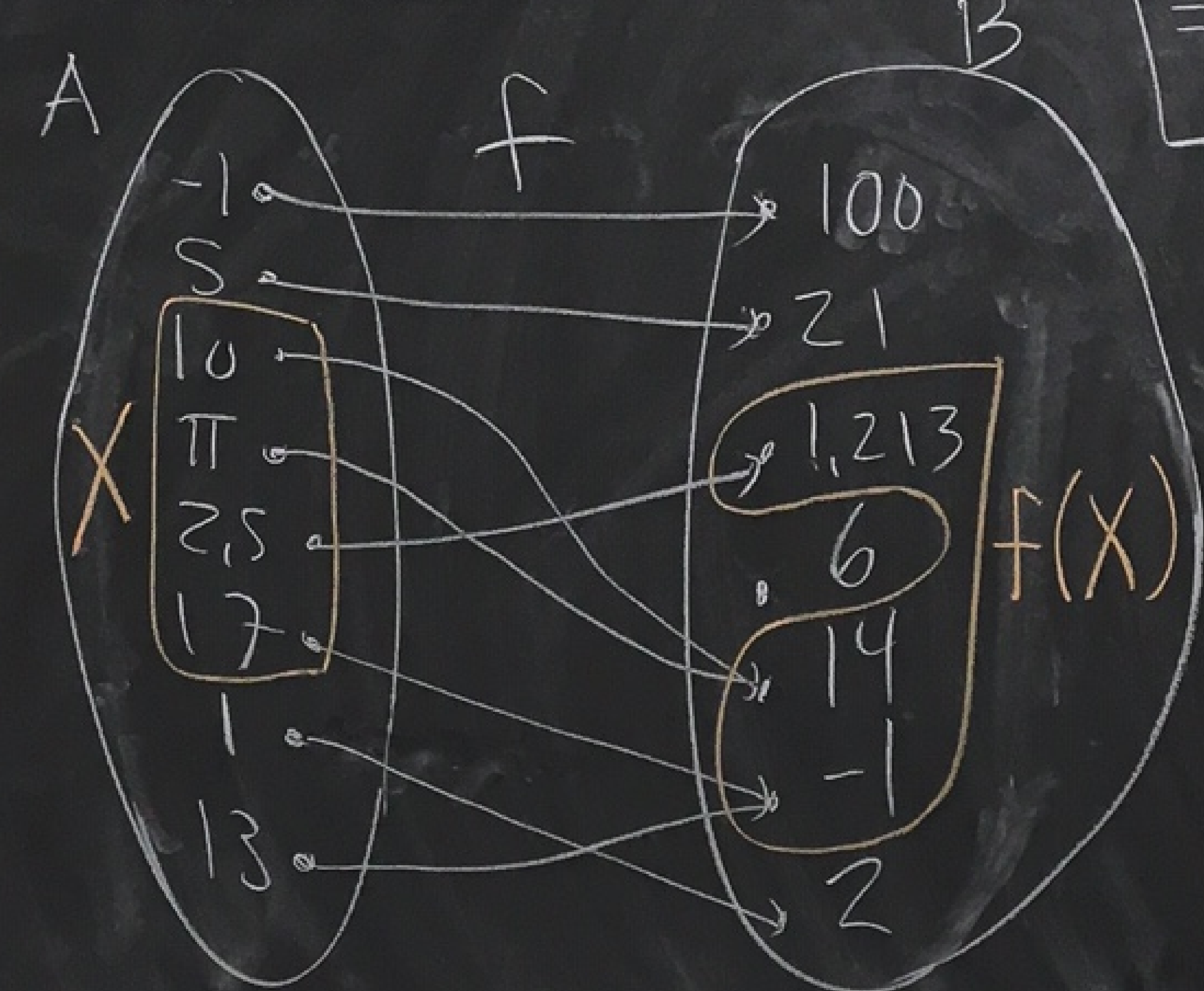


Calculate $f(X)$ where

$X = \{10, \pi, 2.5, 17\}$

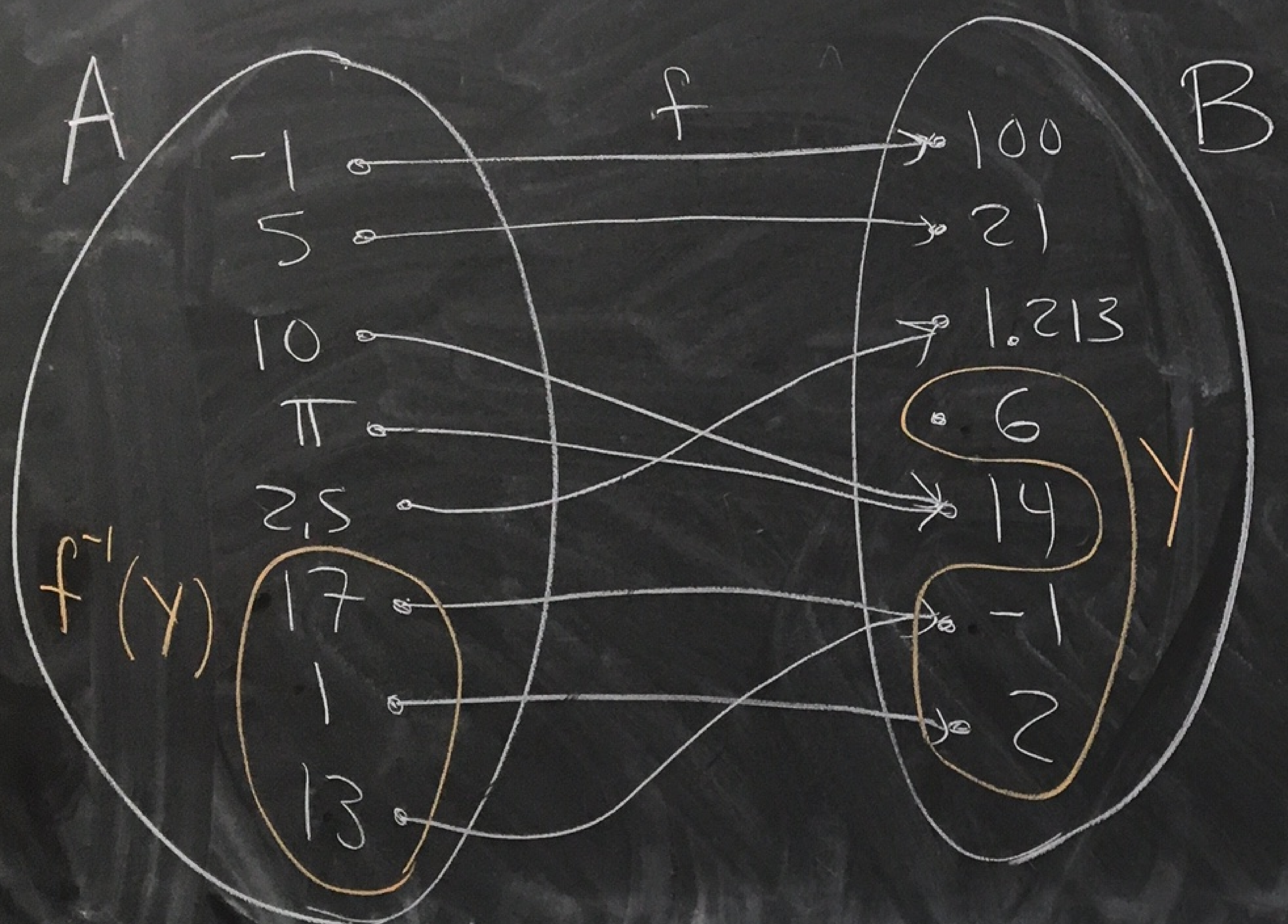
$f(X) = \{f(10), f(\pi), f(2.5), f(17)\} = \{14, 14, 1.213, -1\}$

$= \{14, 1.213, -1\}$



Let $Y = \{6, -1, 2\}$.

Calculate $f^{-1}(Y) = \{a \in A \mid f(a) \in Y\}$



$f(17) = -1 \in Y$
$f(1) = 2 \in Y$
$f(13) = -1 \in Y$

$$f^{-1}(Y) = \{17, 1, 13\}$$

note! This does not mean the inverse function

<u>Thm:</u> Let A, B, W, Z be sets where $W \subseteq A$ and $Z \subseteq A$. Let $f: A \to B$. Then

① $f(W \cap Z) \subseteq f(W) \cap f(Z)$ ← Hammack 12.6 #7

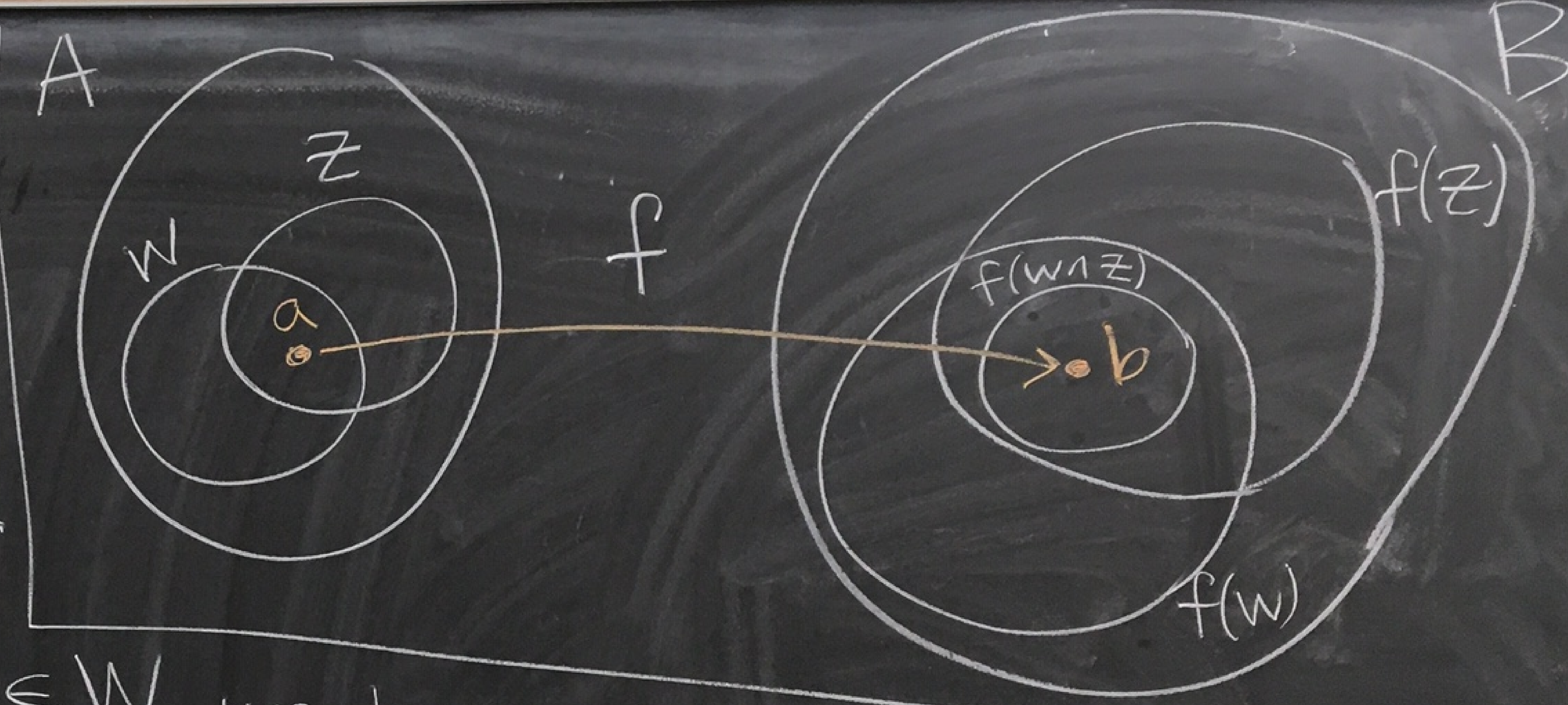
② Give an example to show that $f(W \cap Z) = f(W) \cap f(Z)$ is not always true. ← Hammack 12.6 #8

③ $f(W \cup Z) = f(W) \cup f(Z)$ ← HW 4 #14

1. Let $b \in f(W \cap Z)$.

So, $b = f(a)$ where $a \in W \cap Z$.

So, $a \in W$ and $a \in Z$.



Since $b = f(a)$ and $a \in W$ we know $b \in f(W)$.

Since $b = f(a)$ and $a \in Z$ we know $b \in f(Z)$.

Thus, $b \in f(W) \cap f(Z)$.

Therefore, $f(W \cap Z) \subseteq f(W) \cap f(Z)$.