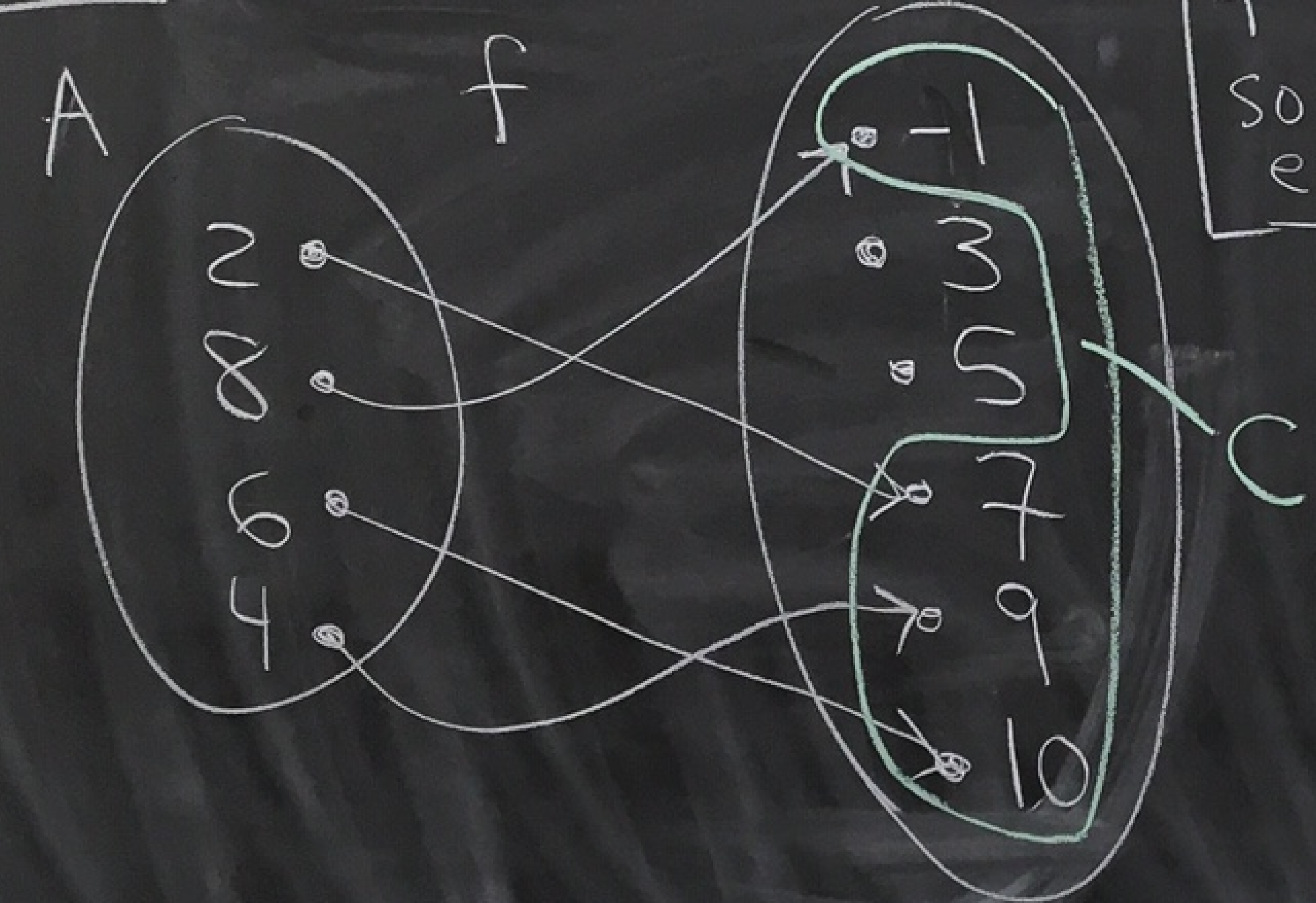
Mon 10/28

Last time recap:

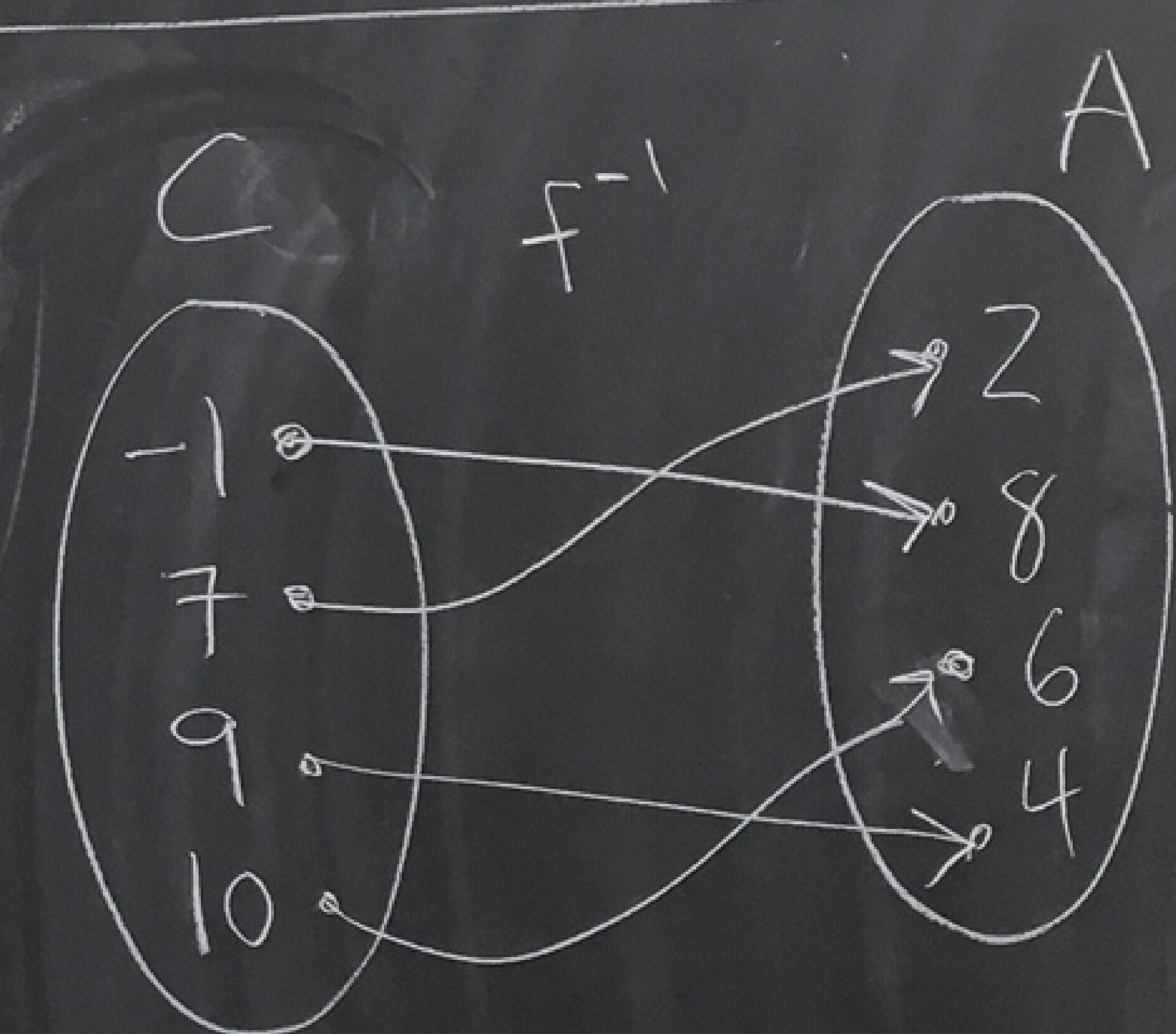
$f: A \to B$ is one-to-one with $C = \text{range}(f)$.

Define $f^{-1}: C \longrightarrow A$ where $f^{-1}(c) = a$ iff $f(a) = c$

Ex:



f is 1-1 so f^{-1} exists



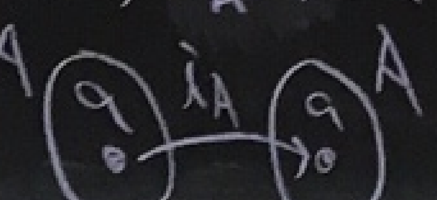
Theorem: Let A, B be sets.
Let $f: A \to B$ be one-to-one.
Let $C = \text{range}(f)$ and $f^{-1}: C \to A$
be the inverse function of f.
Then the following are true:

① $\text{domain}(f^{-1}) = \text{range}(f)$

② $\text{range}(f^{-1}) = \text{domain}(f)$.
In particular, $f^{-1}: C \to A$ is onto A.

③ f^{-1} is one-to-one

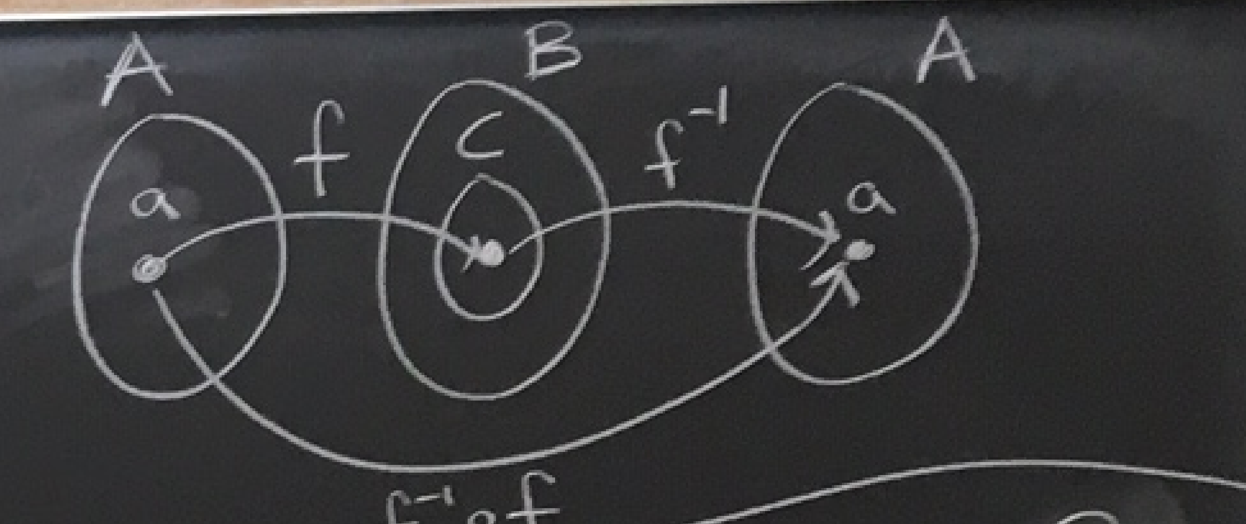
Here $i_A: A \to A$ is the identity function, $i_A(a) = a$ for all $a \in A$.

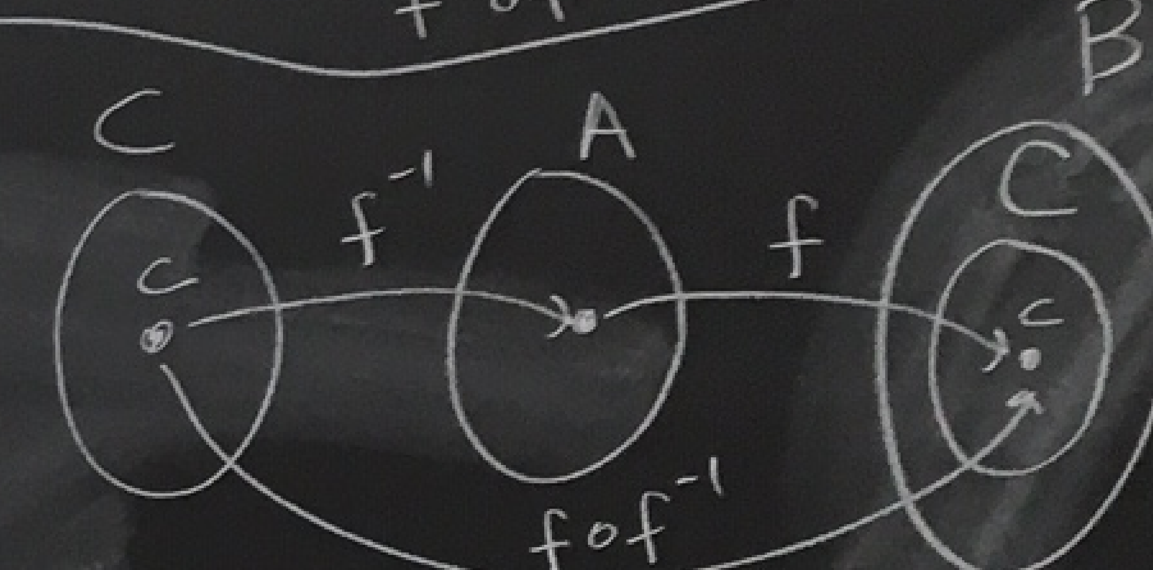


④ $(f^{-1} \circ f)(a) = a$
for all $a \in A$.
So, $f^{-1} \circ f = i_A$.

⑤ $(f \circ f^{-1})(c) = c$
for all $c \in C$.

⑥ If $g: C \to A$
and $g \circ f = i_A$,
then $g = f^{-1}$.





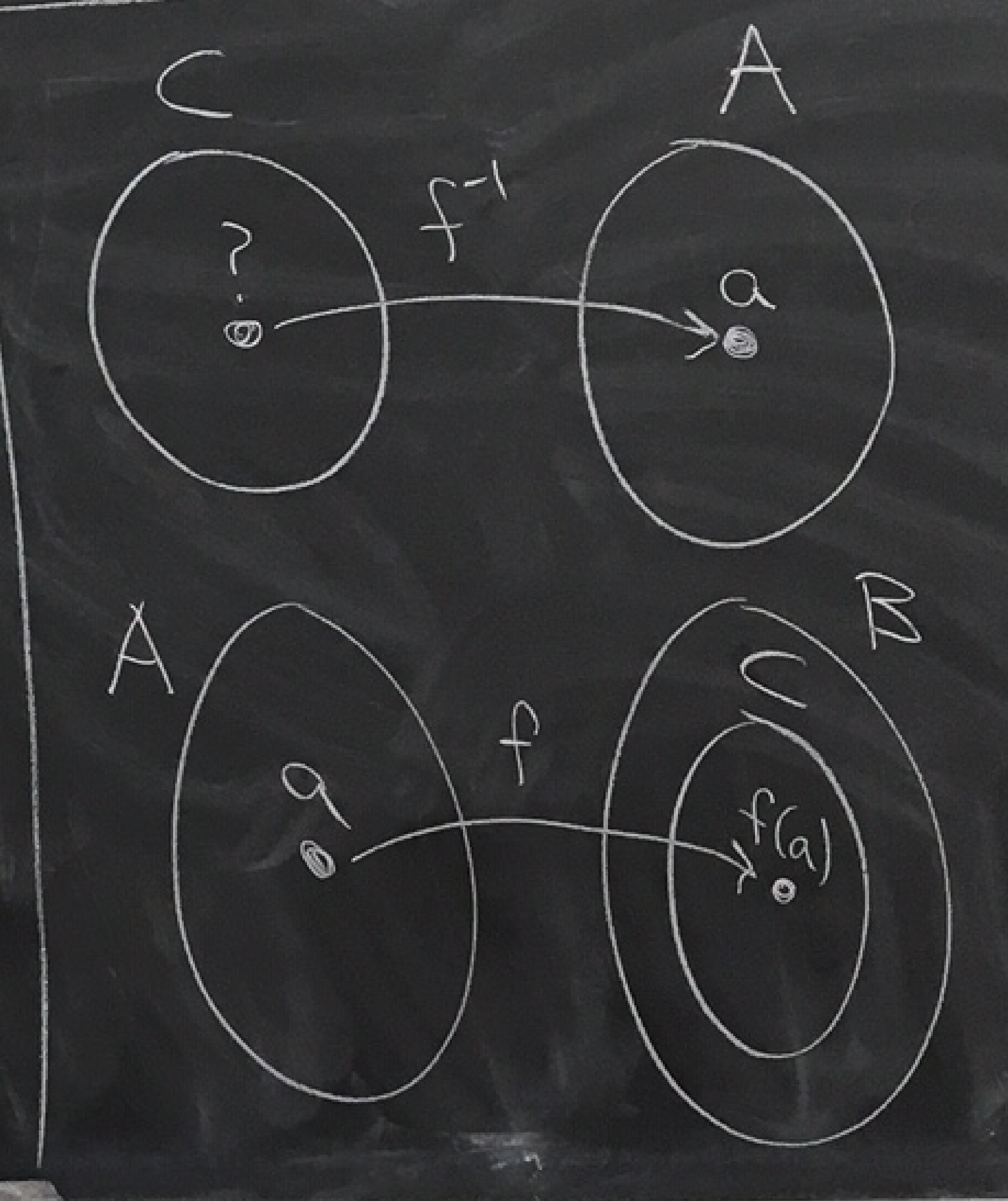
⑥ is a way to verify that some function is f^{-1}

pf:

① This is by def of f^{-1}.
That is, domain$(f^{-1}) = C =$ range(f).

② Let's show that range$(f^{-1}) = A$.
Pick some $a \in A$.
We want $c \in C$ with $f^{-1}(c) = a$.
Set $c = f(a)$.
Then by def of f^{-1}, $f^{-1}(c) = a$.
So, $a \in$ range(f^{-1}).
Thus, f^{-1} is onto A and range$(f^{-1}) = A$.

Picture for ②



③ Suppose $f^{-1}(c_1) = f^{-1}(c_2)$

where $c_1, c_2 \in C$.

We need to show that $c_1 = c_2$.

Let $a = f^{-1}(c_1) = f^{-1}(c_2)$.

Since $a = f^{-1}(c_1)$ we have

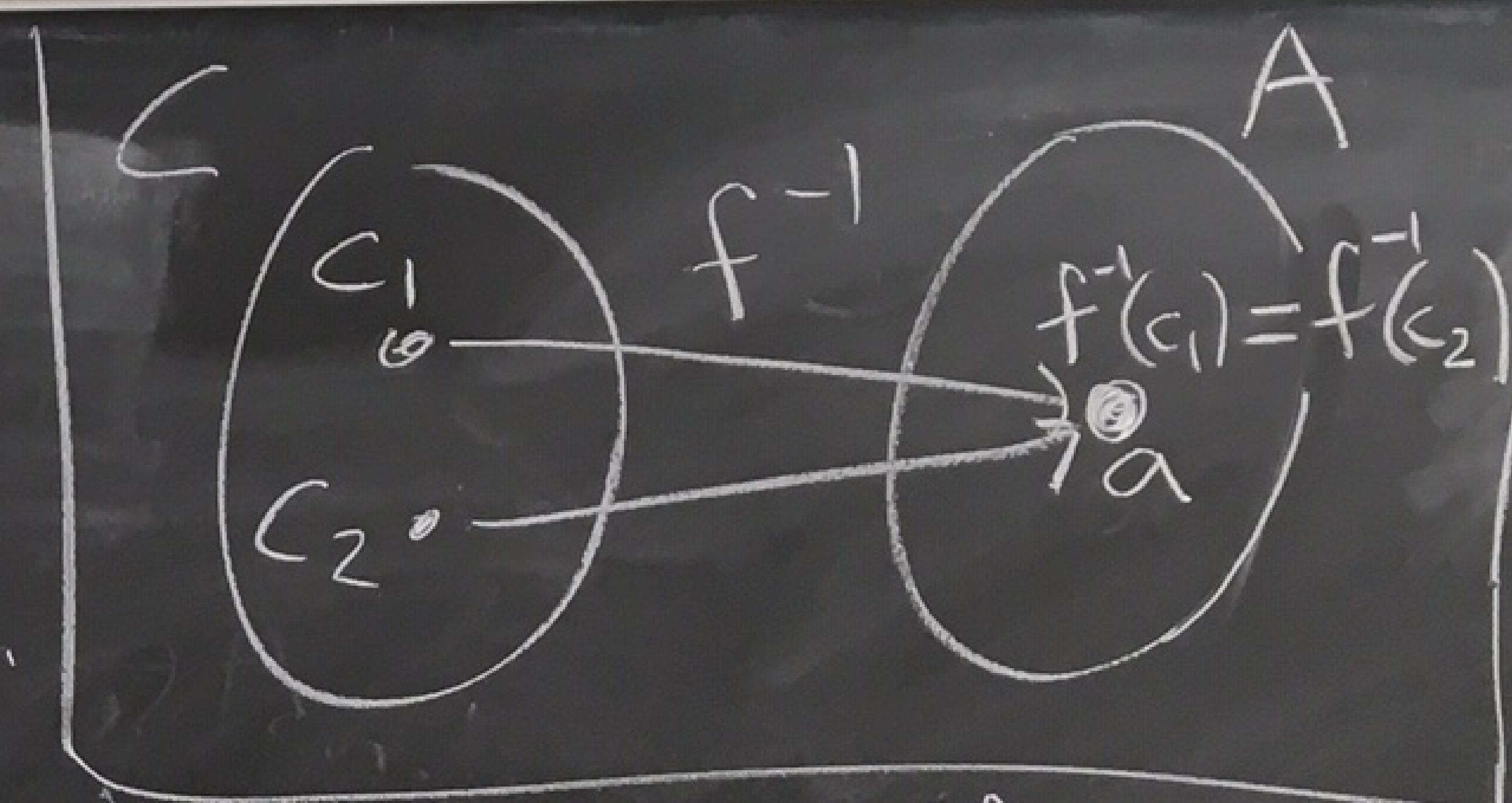
$$f(a) = c_1.$$

Since $a = f^{-1}(c_2)$ we have,

$$f(a) = c_2.$$

So, $c_1 = f(a) = c_2$.

Thus, f^{-1} is one-to-one.



If $c_1 \neq c_2$, then f^{-1} would not be 1-1.
We want to show that this doesn't happen.

④ Let's show that $f^{-1} \circ f = i_A$.

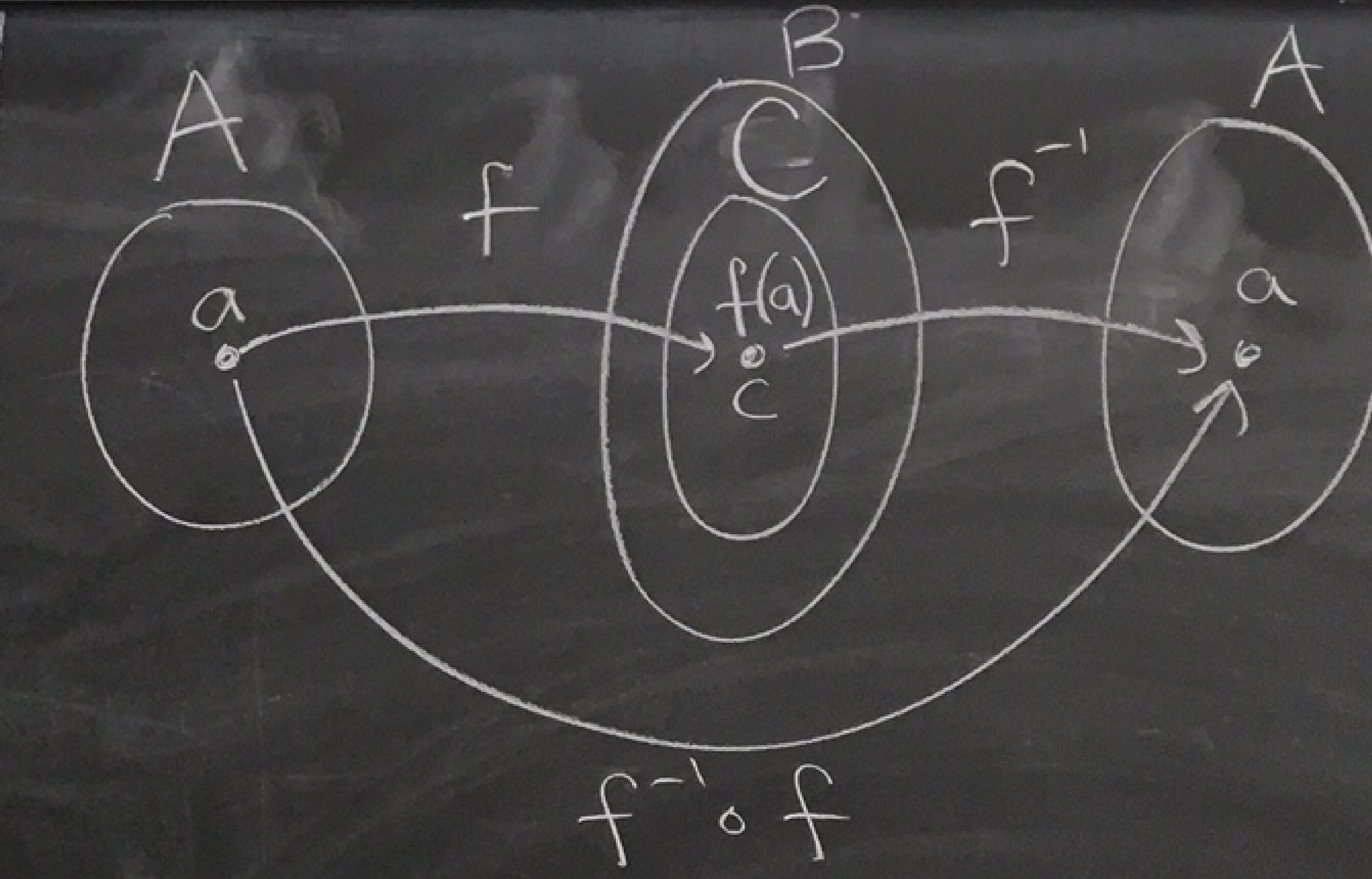
Let $a \in A$.

Set $c = f(a)$.

Then, $f^{-1}(c) = a$ by def of f^{-1}.

So,

$$(f^{-1} \circ f)(a) = f^{-1}(f(a)) = f^{-1}(c) = a = i_A(a)$$

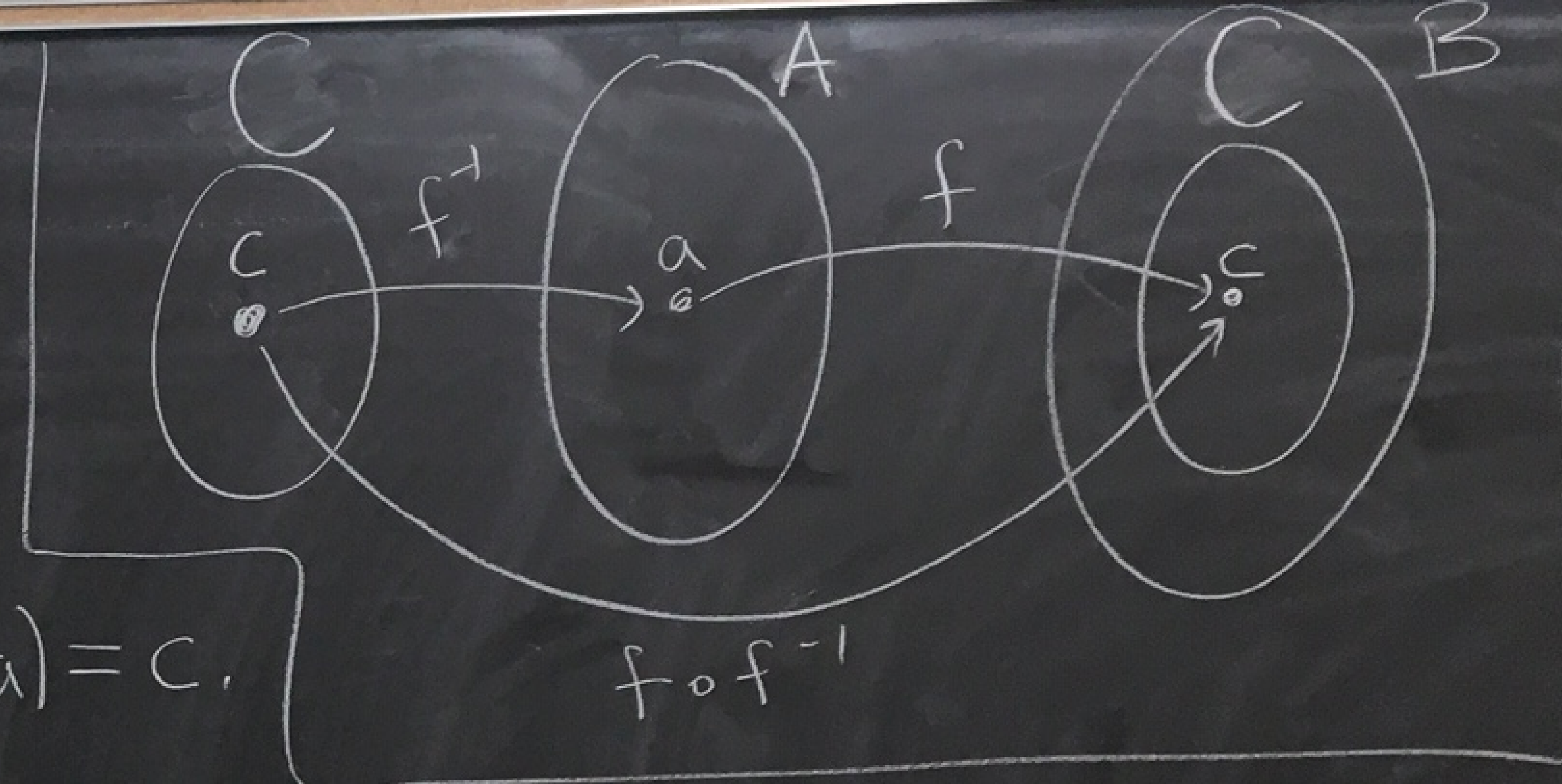


To show that two functions $g: X \to Y$ and $h: X \to Y$ are equal, you show $g(x) = h(x)$ for all $x \in X$.

Since, $(f^{-1} \circ f)(a) = i_A(a)$ for all $a \in A$, we have $f^{-1} \circ f = i_A$.

⑤ Let's show

$(f \circ f^{-1})(c) = c$ for all $c \in C$.



Pick $c \in C$.

Then $f^{-1}(c) = a$ where $a \in A$ and $f(a) = c$.

So, $(f \circ f^{-1})(c) = f(f^{-1}(c)) = f(a) = c$.

(6) Let $g : C \to A$.

Suppose $g \circ f = i_A$.

We need to show that $g = f^{-1}$.

Need to show

$g(c) = f^{-1}(c)$

for all $c \in C$

Pick some $c \in C$.

Then $f^{-1}(c) = a$ where $a \in A$ and $f(a) = c$.

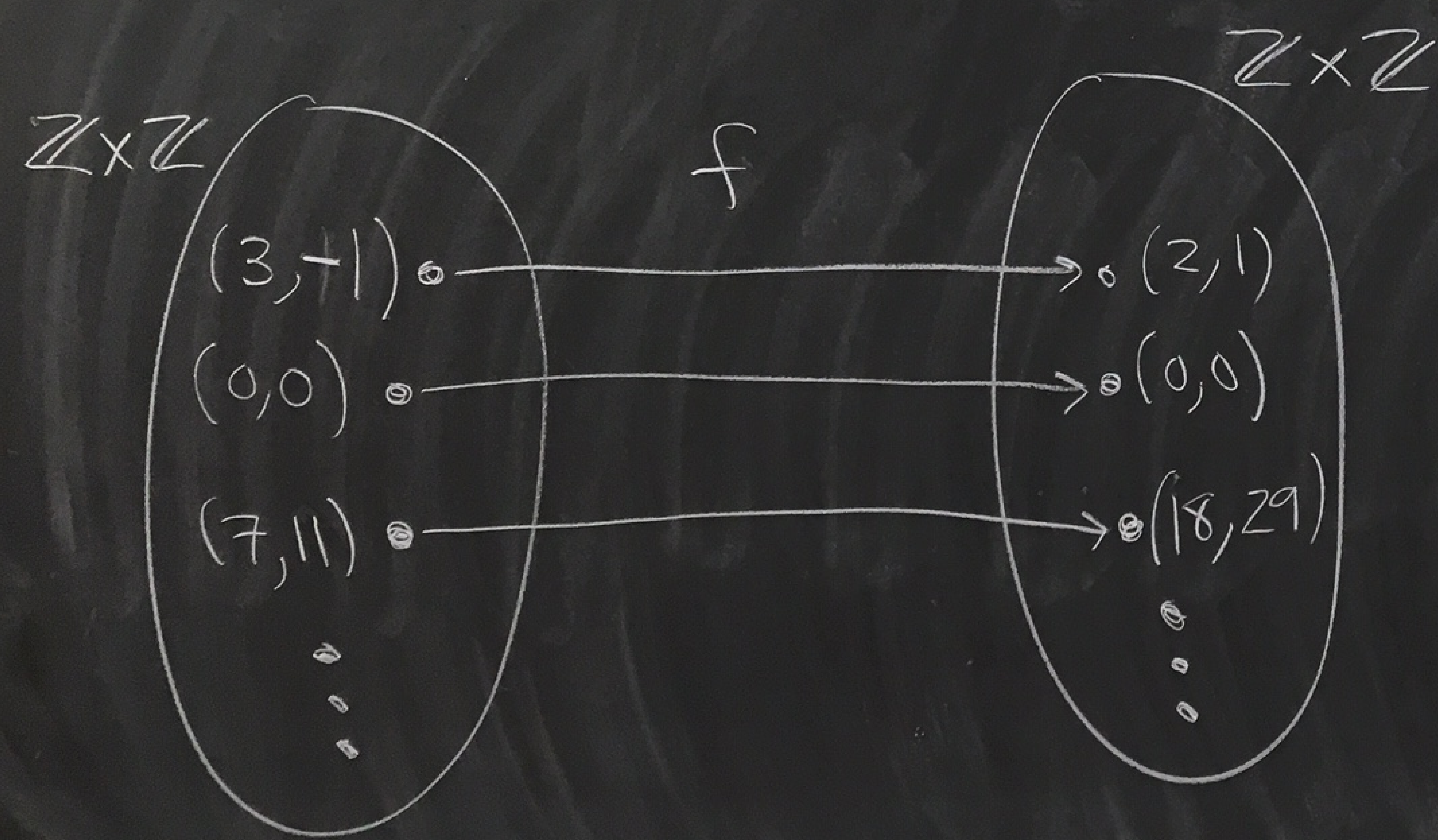
So,

$$g(c) = g(f(a)) = (g \circ f)(a) = i_A(a) = a = f^{-1}(c).$$

That is, $g(c) = f^{-1}(c)$.

Since c was arbitrary we have $g = f^{-1}$. $\blacksquare$

Ex: Consider $f: \mathbb{Z}\times\mathbb{Z} \to \mathbb{Z}\times\mathbb{Z}$ given by $f(m,n) = (m+n,\ m+2n)$



$$f(7,11) = (7+11,\ 7+2\cdot 11)$$
$$= (18, 29)$$
$$f(3,-1) = (3-1,\ 3+(2)(-1))$$
$$= (2,1)$$

f is one-to-one

pf: Suppose $f(m_1, n_1) = f(m_2, n_2)$
for some $(m_1, n_1), (m_2, n_2) \in \mathbb{Z} \times \mathbb{Z}$.
We need to show that $(m_1, n_1) = (m_2, n_2)$.
Since $f(m_1, n_1) = f(m_2, n_2)$ we know that

$$(m_1 + n_1, m_1 + 2n_1) = (m_2 + n_2, m_2 + 2n_2).$$

So,

$$\left.\begin{aligned} m_1 + n_1 &= m_2 + n_2 \\ m_1 + 2n_1 &= m_2 + 2n_2 \end{aligned}\right\} \begin{aligned} &\text{eq } 1 \\ &\text{eq } 2 \end{aligned}$$

Add $-(\text{eq} ①)$ to eq ② to get $n_1 = n_2$.

$$-(m_1+n_1) = -(m_2+n_2)$$
$$+\ (m_1+2n_1) = (m_2+2n_2)$$
$$n_1 = n_2$$

Replace $n_1 = n_2$ into eq ① to get

$$m_1 + n_1 = m_2 + n_1.$$

So, $m_1 = m_2$.

Thus, $(m_1, n_1) = (m_2, n_2)$.

So, f is one-to-one. $\blacksquare$