Oct 21
Monday

HW 3

⑧ $S = \mathbb{N} \times \mathbb{N}$

$(a,b) \sim (c,d)$ iff $a+d = b+c$

This is an equivalence relation by part (c). We proved this in class.

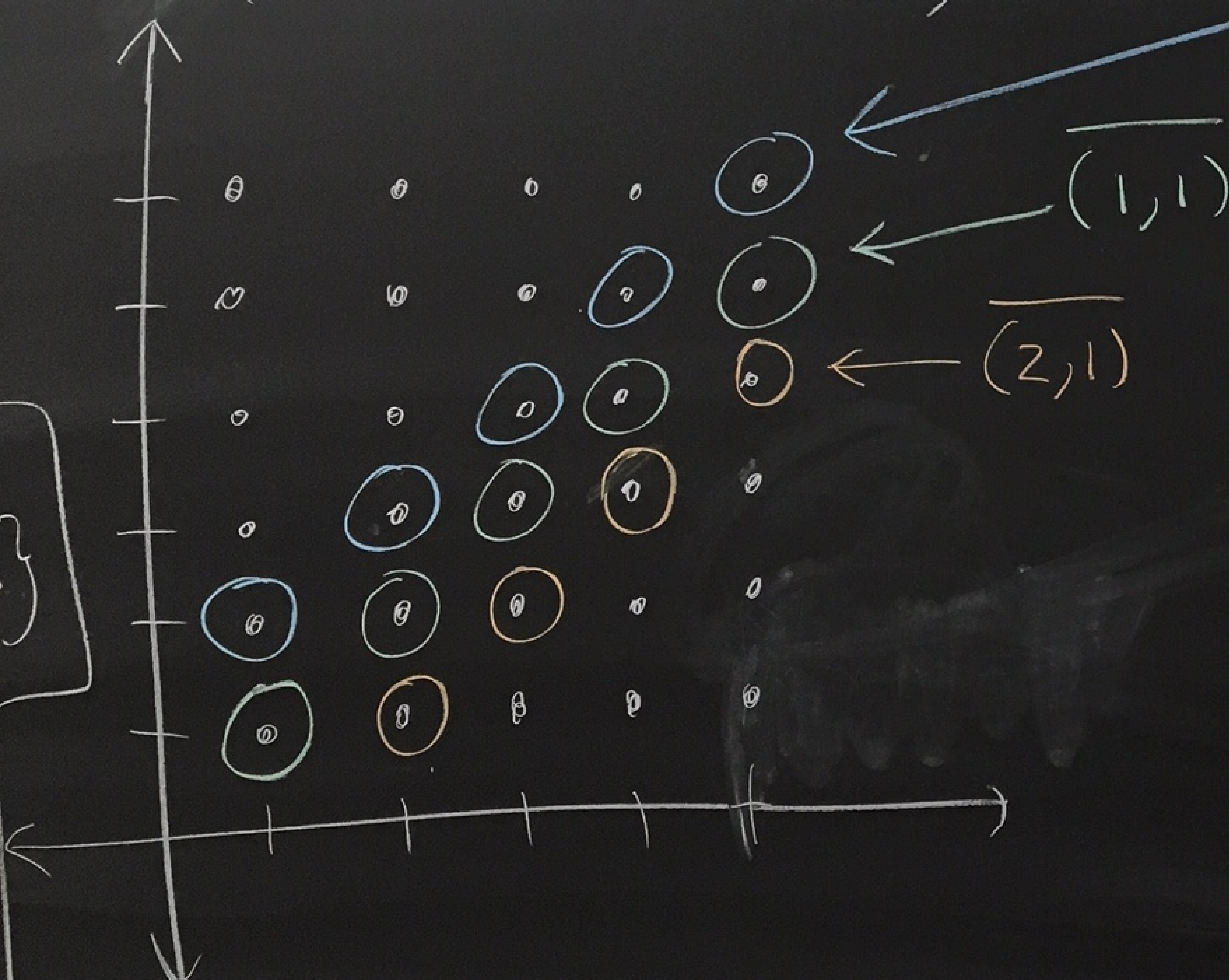
$(2,1) \sim (4,3)$

since

$2+3 = 1+4$

$\overline{(2,1)} = \{(2,1), (3,2), (4,3), \ldots\}$

(picture of equivalence classes)



(e) Define

$$\overline{(a,b)} \oplus \overline{(c,d)} = \overline{(a+c, b+d)}$$

We want to prove that $\oplus$ is well-defined on the set of equivalence classes $\mathbb{N}\times\mathbb{N}/\sim$

Ex: $\overline{(1,2)} \oplus \overline{(1,1)} = \overline{(1+1, 2+1)} = \overline{(2,3)}$

$\updownarrow$ equal $\quad \updownarrow$ equal $\quad \updownarrow$ equal

$\overline{(4,5)} \oplus \overline{(3,3)} = \overline{(4+3, 5+3)} = \overline{(7,8)}$

(2)

proof of (e):

① Pick $\overline{(a,b)}, \overline{(c,d)} \in \mathbb{N}\times\mathbb{N}/\sim$ where $a,b,c,d \in \mathbb{N} = \{1,2,3,4,\dots\}$

Then $a+c \in \mathbb{N}$ and $b+d \in \mathbb{N}$.

So, $\overline{(a,b)} \oplus \overline{(c,d)} = \overline{(a+c, b+d)} \in \mathbb{N}\times\mathbb{N}/\sim$.

② Let $\overline{(a_1,b_1)}, \overline{(a_2,b_2)}, \overline{(c_1,d_1)}, \overline{(c_2,d_2)} \in \mathbb{N}\times\mathbb{N}/\sim$

and $\overline{(a_1,b_1)} = \overline{(a_2,b_2)}$ and $\overline{(c_1,d_1)} = \overline{(c_2,d_2)}$.

We need to show that

$$\overline{(a_1,b_1)} \oplus \overline{(c_1,d_1)} = \overline{(a_2,b_2)} \oplus \overline{(c_2,d_2)}.$$

Since $\overline{(a_1,b_1)} = \overline{(a_2,b_2)}$ we have that $a_1+b_2 = b_1+a_2$.

Since $\overline{(c_1,d_1)} = \overline{(c_2,d_2)}$ we have that $c_1+d_2 = d_1+c_2$.

Adding the above two equations gives $a_1+c_1+b_2+d_2 = b_1+d_1+a_2+c_2$.

Scratchwork

Want to show:

$$\overline{(a_1+c_1,\, b_1+d_1)} = \overline{(a_2+c_2,\, b_2+d_2)}$$

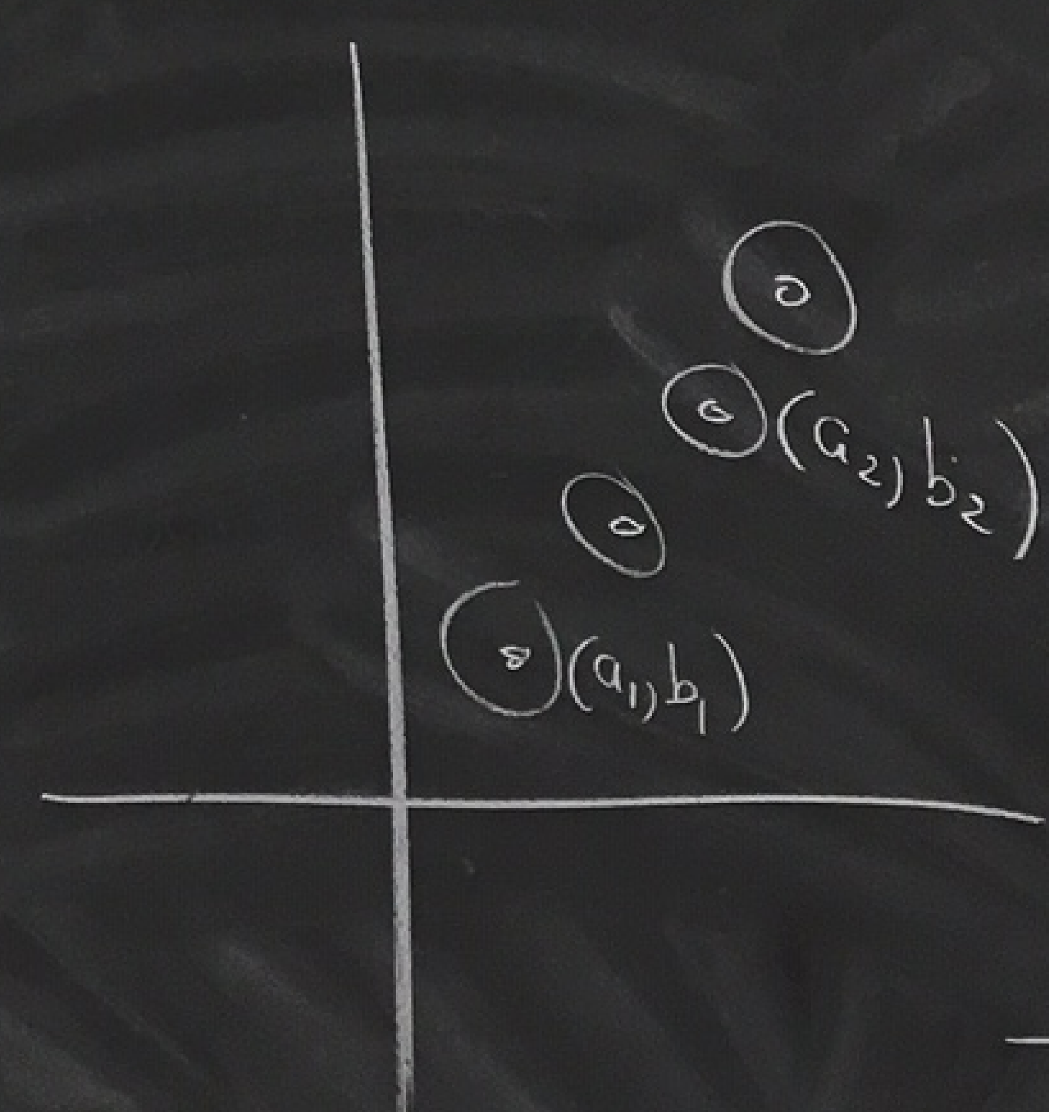
$$a_1+c_1+b_2+d_2 = b_1+d_1+a_2+c_2$$

So, $\overline{(a_1+c_1, b_1+d_1)} = \overline{(a_2+c_2, b_2+d_2)}$.

That is, $\overline{(a_1, b_1)} \oplus \overline{(c_1, d_1)} = \overline{(a_2, b_2)} \oplus \overline{(c_2, d_2)}$.

$\blacksquare$

$\boxed{\overline{x} = \overline{y} \text{ iff } x \sim y}$

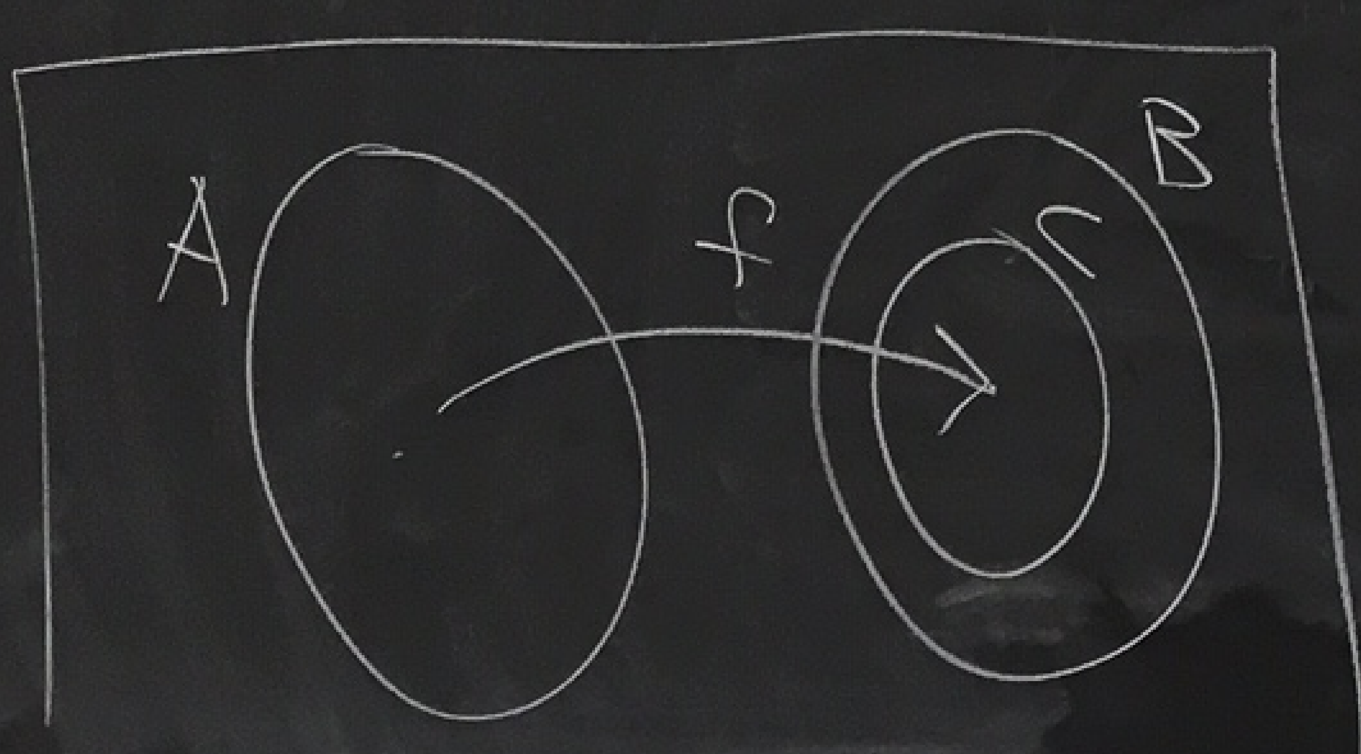


$\overline{(a_1, b_1)} = \overline{(a_2, b_2)}$

$\longrightarrow (a_1, b_1) \sim (a_2, b_2)$

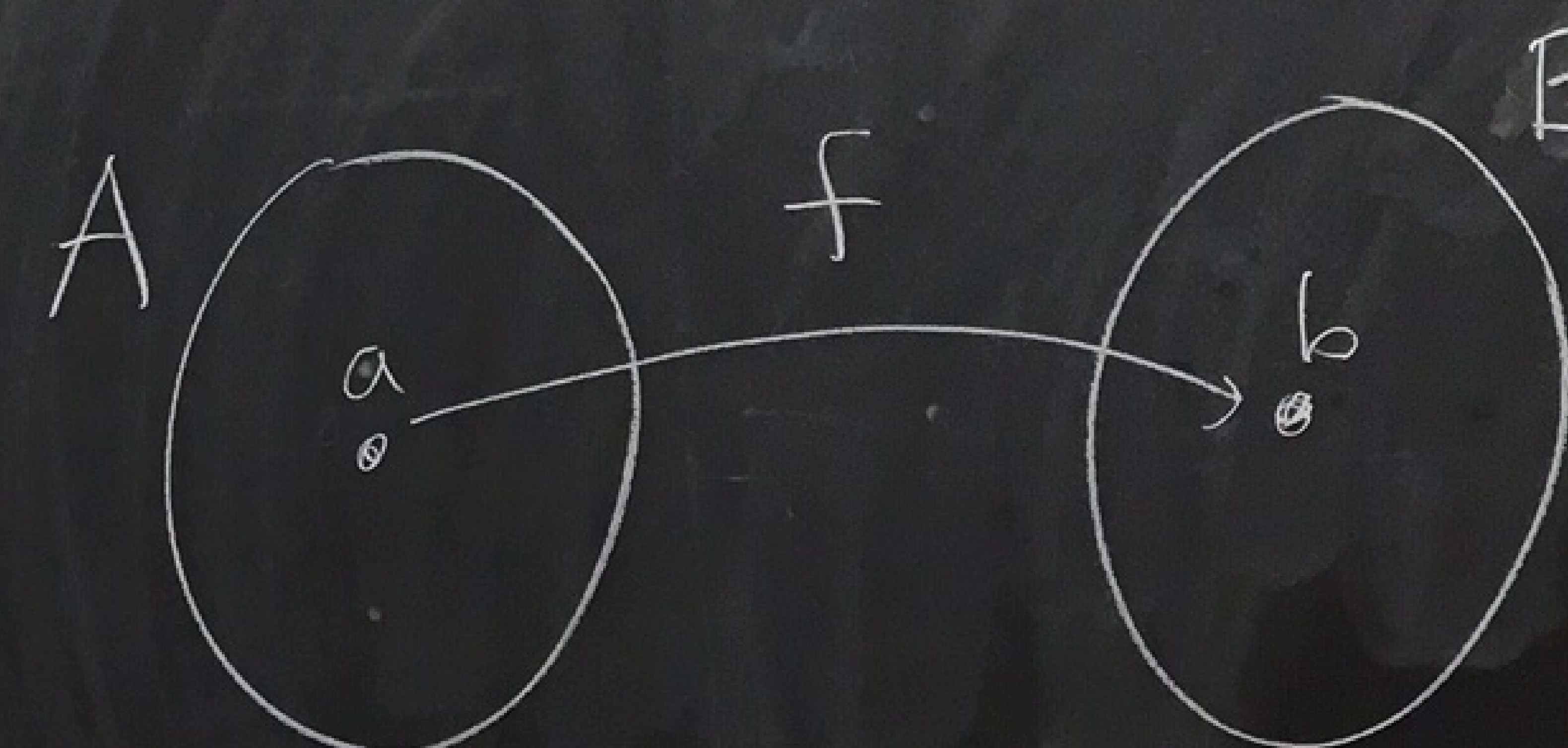
$\longrightarrow a_1 + b_2 = b_1 + a_2$

__Def:__ Let A and B be sets and $f: A \to B$. Let C be the range of f. We say that f is __surjective__ or __onto B__ if $C = B$.



Another way to say this:

f is __onto B__ if for every $b \in B$ there exists $a \in A$ with $f(a) = b$



There can be more than one a with $f(a) = b$.

Scratchwork

$b = f(a) = 2a - 5$

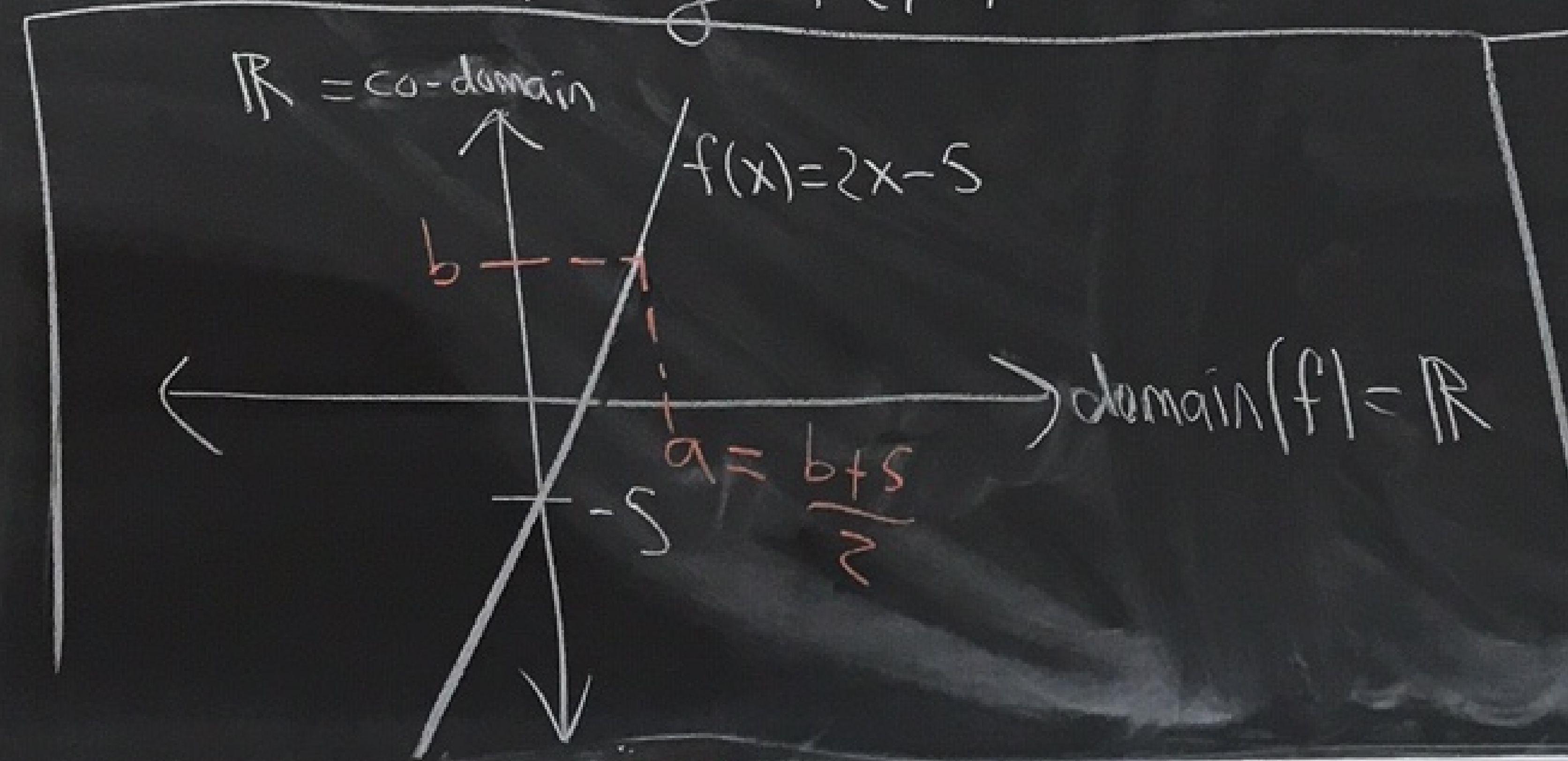
$\frac{b+5}{2} = a$

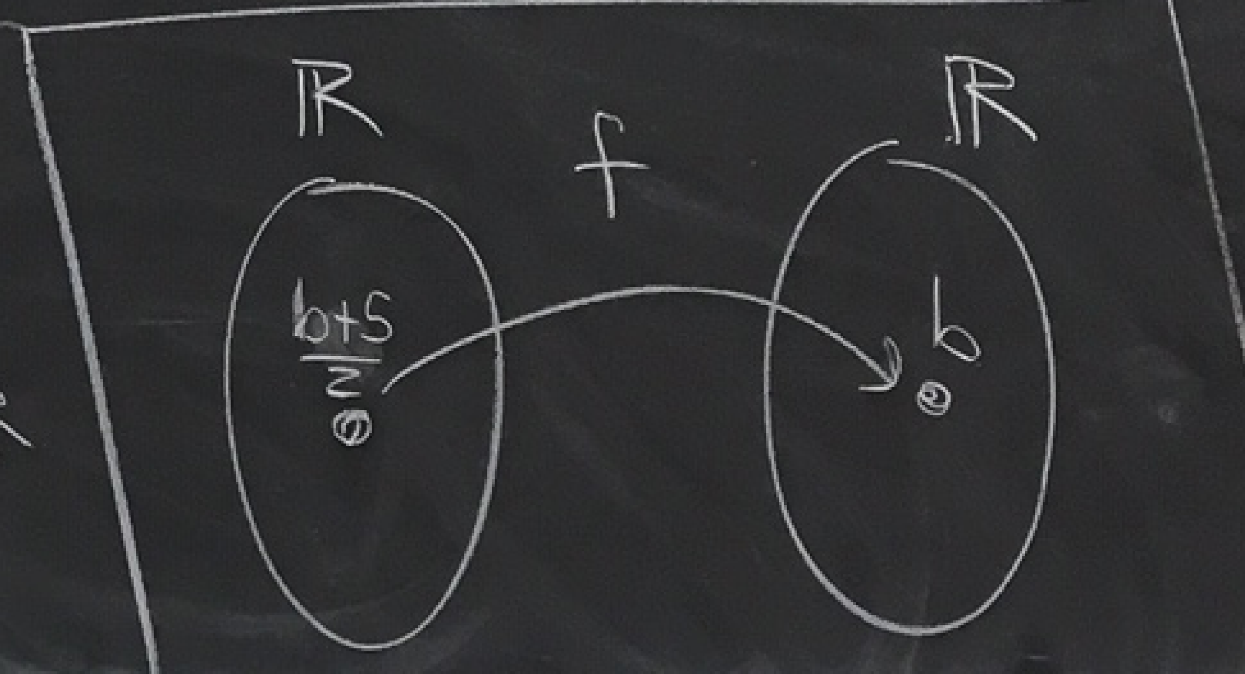
How to prove $f: A \to B$ is onto

Pick/Let $b \in B$.

Find $a \in A$ with $f(a) = b$.

Ex: Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x - 5$.





Proof that f is onto

Let $b \in \mathbb{R}$.

Set $a = \frac{b+5}{2}$.

Then $a \in \mathbb{R}$ and

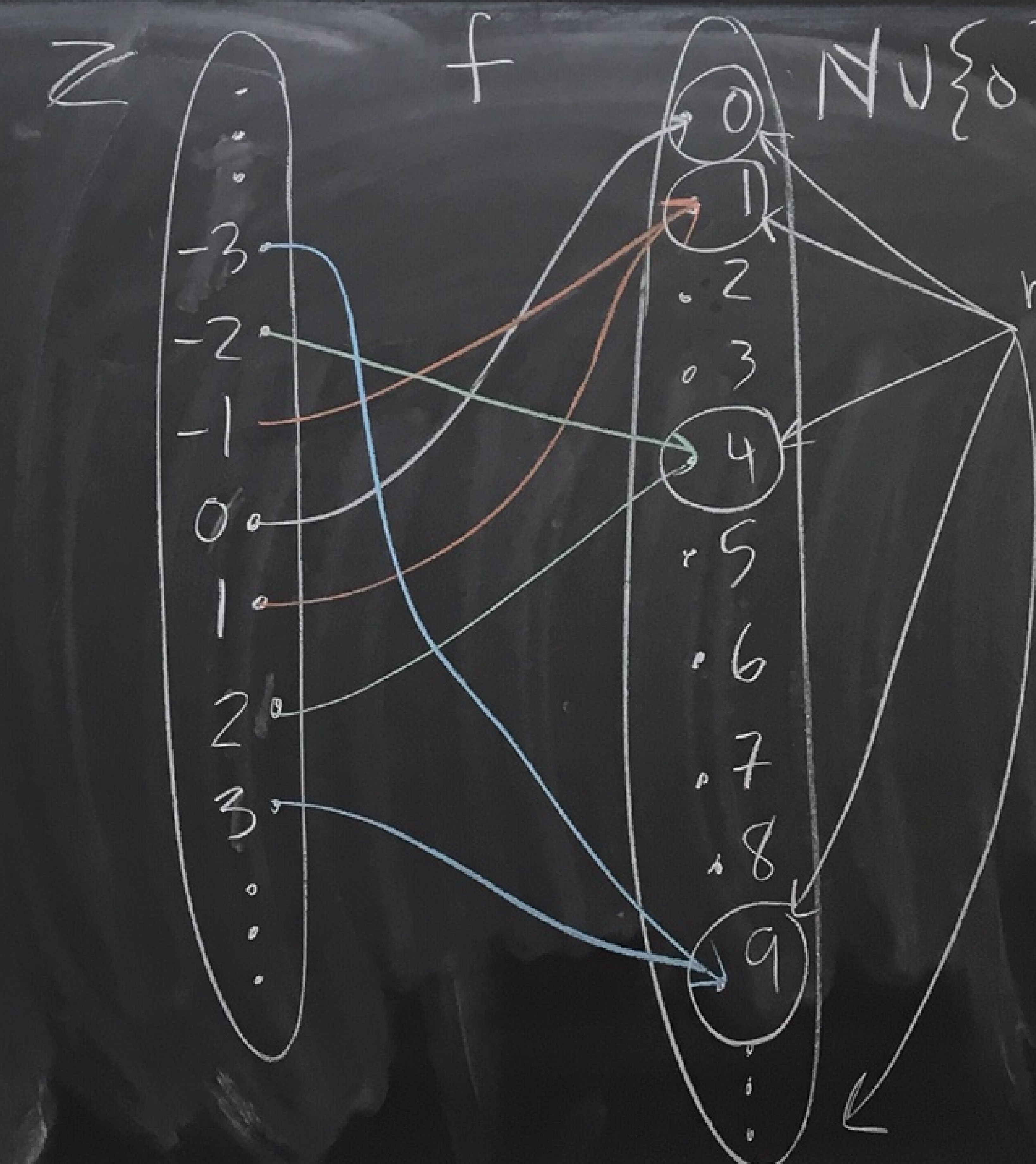
$$f(a) = 2a - 5 = 2\left(\frac{b+5}{2}\right) - 5 = b.$$

Since b was arbitary, f is onto. ∎

How to show $f: A \to B$ is not onto

Find some $b \in B$ where there does not exist $a \in A$ with $f(a) = b$.

Ex: Let $f: \mathbb{Z} \to \mathbb{N} \cup \{0\}$ be given by $f(x) = x^2$.



range(f) $\neq \mathbb{N} \cup \{0\}$.
So f is not onto.

pf that f is not onto:

Consider $2 \in \mathbb{N} \cup \{0\}$.

There is no $x \in \mathbb{Z}$ with $f(x) = 2$.

Why?

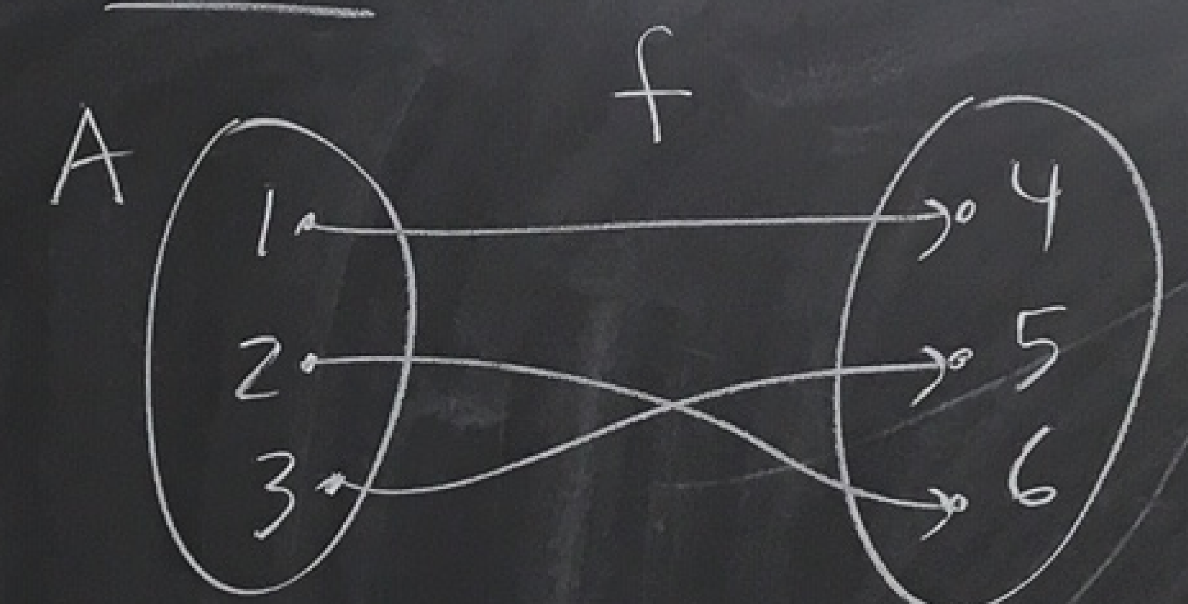
If $x^2 = 2$, then $x = \pm\sqrt{2} \notin \mathbb{Z}$.

That's why.

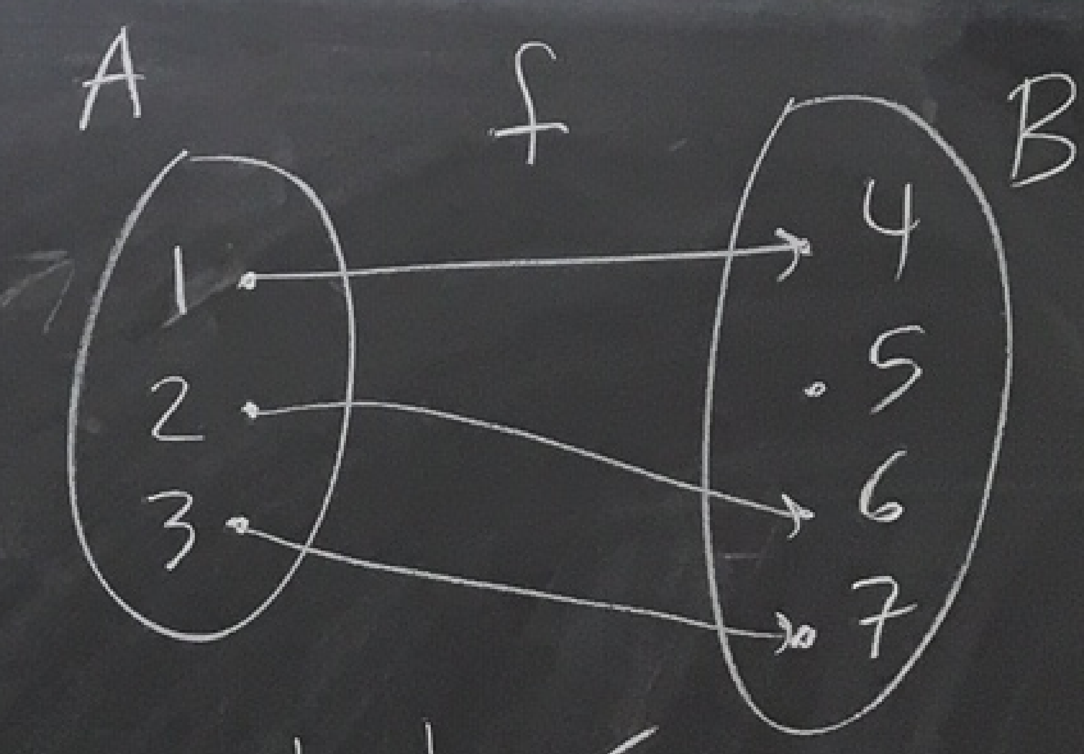
$\blacksquare$

Def: Let A and B be sets and $f: A \to B$. We say that f is a **bijection** if f is one-to-one and onto.

Ex:



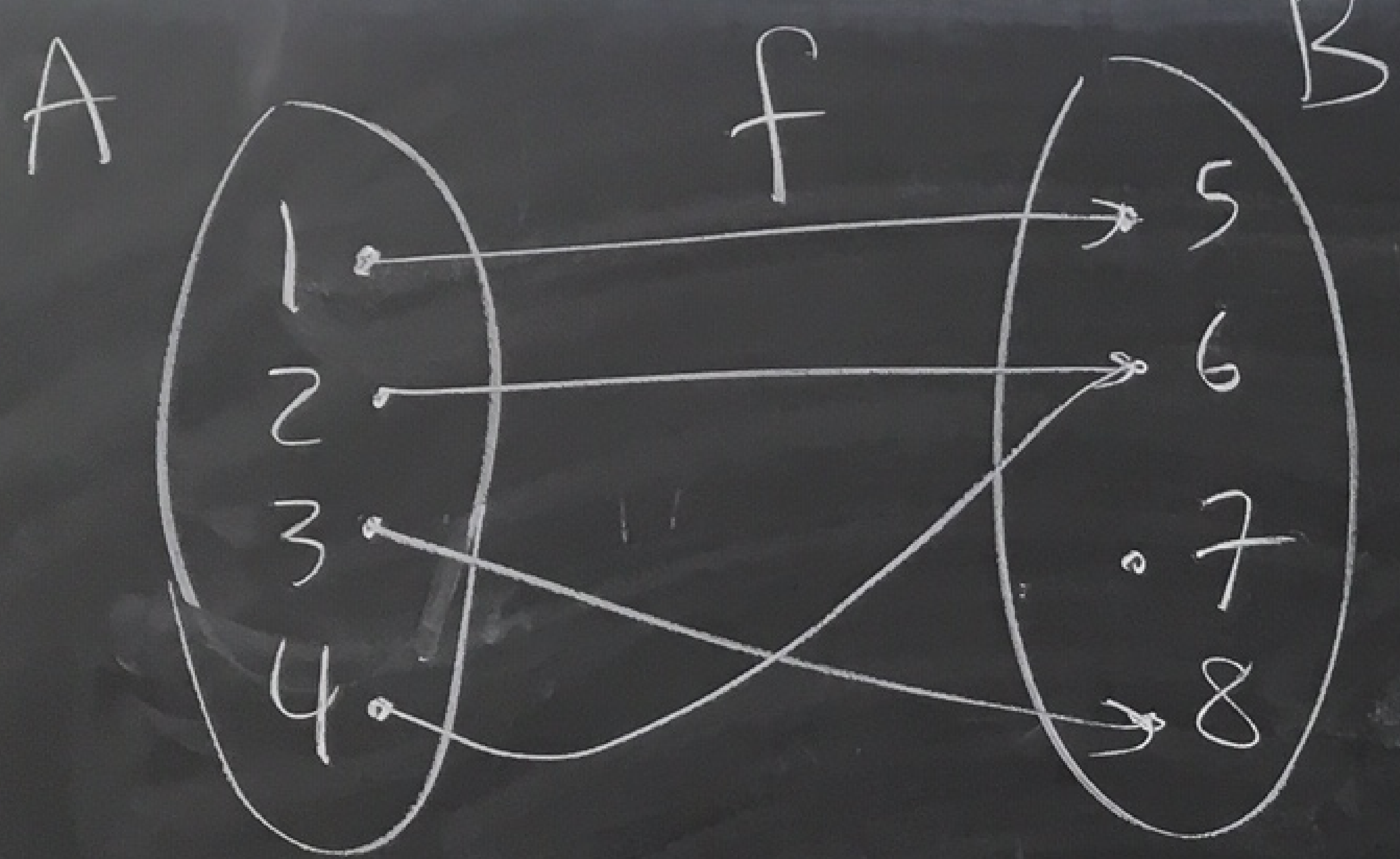
1-1 and onto
bijection



1-1 ✓

onto ✗ ← $5 \notin \text{range}(f)$

not a bijection

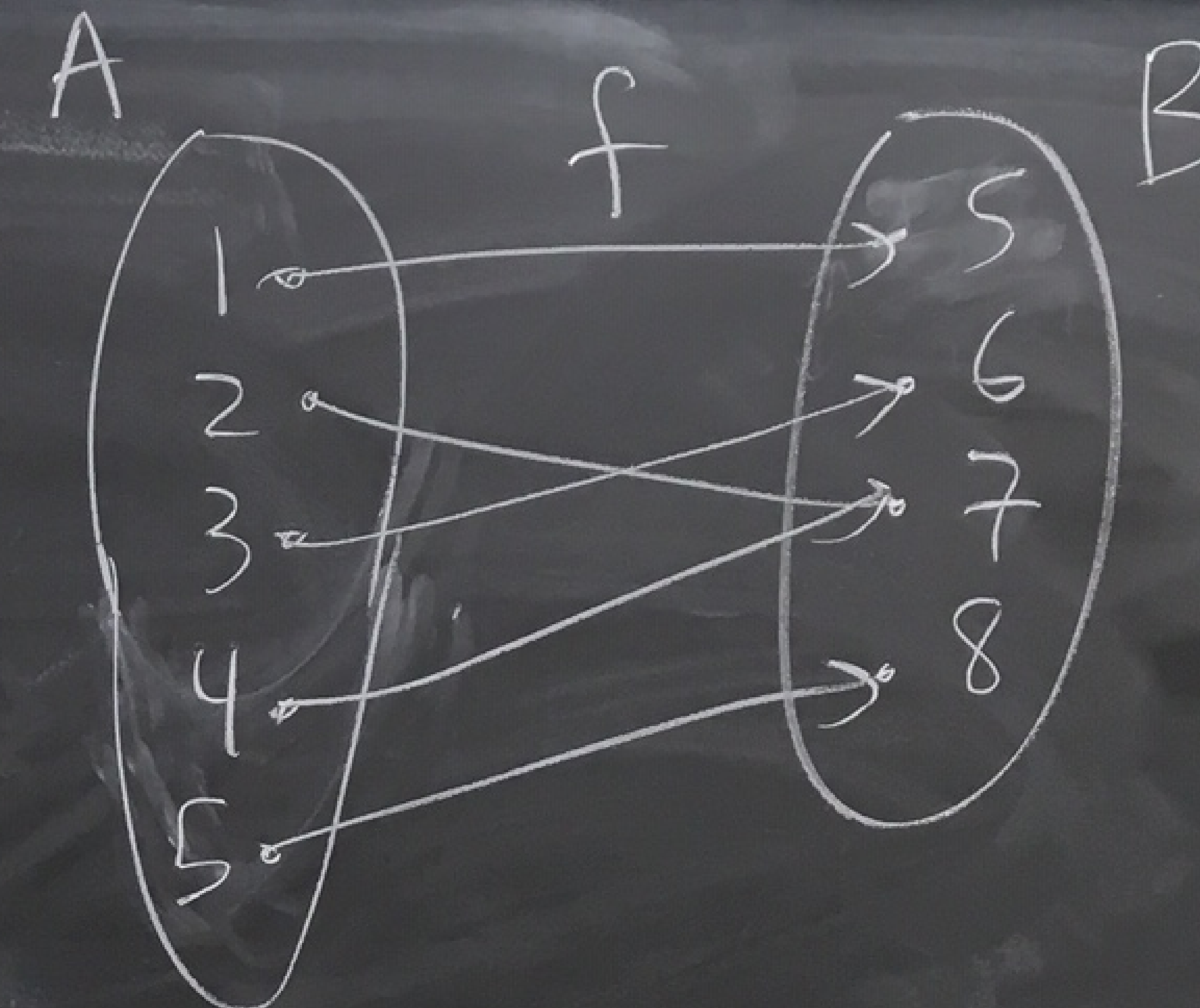


1-1 ✗ $f(2)=f(4)$

onto ✗ $7 \notin \text{range}(f)$

bijection ✗

$a = \frac{b+5}{2}$



1-1 ✗ ← ($f(2)=f(4)$)

onto ✓

bijection ✗