10/2
Weds

Def: A <u>partition</u> of a set S is a family of sets $\mathcal{A}$ where

① Every $A \in \mathcal{A}$ satisfies $A \subseteq S$.

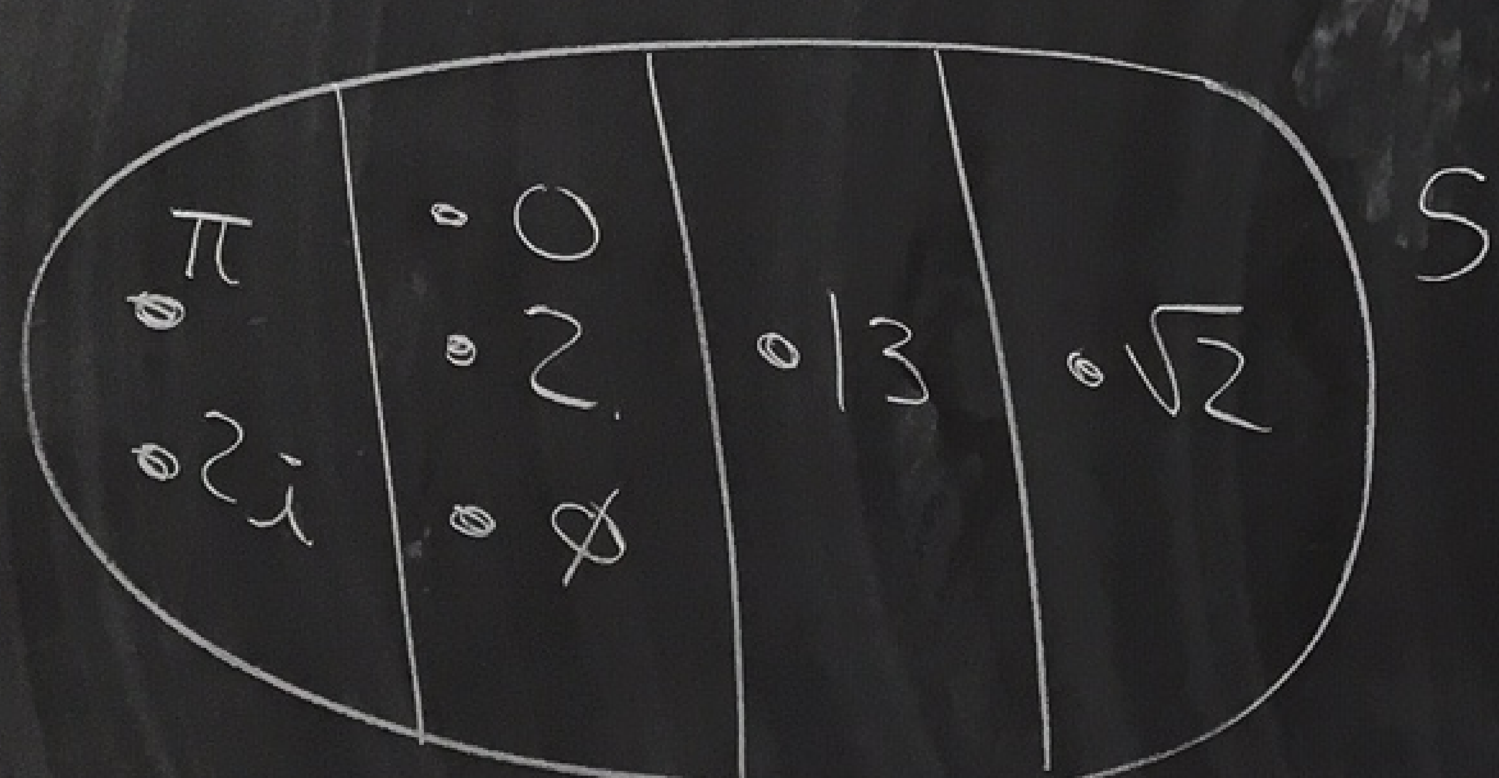
② $\bigcup_{A \in \mathcal{A}} A = S$

③ If $A, B \in \mathcal{A}$ and $A \neq B$ then $A \cap B = \emptyset$

Ex: $S = \{\pi, 13, 0, 2, \sqrt{2}, 2i, \phi\}$

$\mathcal{A} = \{ \{\pi, 2i\}, \{0, 2, \phi\}, \{13\}, \{\sqrt{2}\} \}$

$\mathcal{A}$ partitions S into 4 disjoint pieces.



$\mathcal{A}$ is a partition of S

1. $\{\pi, 2i\} \subseteq S$
   $\{0, 2, \phi\} \subseteq S$
   $\{13\} \subseteq S$
   $\{\sqrt{2}\} \subseteq S$

2. $\bigcup_{A \in \mathcal{A}} A = \{\pi, 2i\} \cup \{0, 2, \phi\} \cup \{13\} \cup \{\sqrt{2}\} = S$

3. $\{\pi, 2i\} \cap \{0, 2, \phi\} = \phi$
   $\{\pi, 2i\} \cap \{13\} = \phi$
   $\{\pi, 2i\} \cap \{\sqrt{2}\} = \phi$
   $\{0, 2, \phi\} \cap \{13\} = \phi$
   $\{0, 2, \phi\} \cap \{\sqrt{2}\} = \phi$
   $\{13\} \cap \{\sqrt{2}\} = \phi$

Ex: $S = \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

$\mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$

$\mathbb{Z}_3$ is a partition of $\mathbb{Z}$

$\overline{0} = \{\ldots, -6, -3, 0, 3, 6, \ldots\}$

$\overline{1} = \{\ldots, -5, -2, 1, 4, 7, \ldots\}$

$\overline{2} = \{\ldots, -4, -1, 2, 5, 8, \ldots\}$

Theorem: Let S be a non-empty set.
Let $\sim$ be an equivalence relation on S.
The set of equivalence classes

$$S/\sim \; = \{ \bar{a} \mid a \in S \}$$

is a partition of S.

Ex: $\mathbb{Z}_4 = \mathbb{Z}/(\equiv \bmod 4) = \{ \bar{a} \mid a \in \mathbb{Z} \} = \{ \bar{0}, \bar{1}, \bar{2}, \bar{3} \}$

Ex:

$S = \mathbb{Z}$

$\sim$ is $\equiv \bmod 3$

$$\bigcup_{a \in \mathbb{Z}} \overline{a} = \overline{0} \cup \overline{1} \cup \overline{-1} \cup \overline{2} \cup \overline{-2} \cup \overline{3} \cup \overline{-3} \cup \ldots$$

$$= \overline{0} \cup \overline{1} \cup \overline{2}$$

$$= \bigcup_{\overline{a} \in \mathbb{Z}_3} \overline{a}$$

proof of theorem:

① Let $\overline{a} \in S/\sim$ where $a \in S$.

Then

$$\overline{a} = \{ b \mid b \in S \text{ and } a \sim b \} \subseteq S.$$

② Recall that if $\overline{a} \in S/\sim$ then $a \in \overline{a}$ by the super-equivalence class theorem.

Thus, $S = \bigcup_{a \in S} \{a\} \subseteq \bigcup_{a \in S} \overline{a} \subseteq \bigcup_{\overline{a} \in S/\sim} \overline{a}$

$\boxed{\{a\} \subseteq \overline{a}}$

So, $S \subseteq \bigcup_{\bar{a} \in S/\sim} \bar{a}$.

From ① each $\bar{a} \subseteq S$, so

$$\bigcup_{\bar{a} \in S/\sim} \bar{a} \subseteq S.$$

Thus,

$$S = \bigcup_{\bar{a} \in S/\sim} \bar{a}$$

③ By the super-duper equivalence class theorem, if $a, b \in S$ then either $\bar{a} = \bar{b}$ or $\bar{a} \cap \bar{b} = \phi$.

$\blacksquare$

Theorem: Let S be a non-empty set. Let $\mathcal{A}$ be a partition of S. Define a relation on S by the following: Given $a, b \in S$, then $a \sim b$ iff there exists $A \in \mathcal{A}$ with $a \in A$ and $b \in A$.

Then:

1. $\sim$ is an equivalence relation on S.
2. $S/\sim \; = \mathcal{A}$

Ex: $S=\{1,2,3,4,5\}$

$$\mathcal{A}=\{\{1,2,3\},\{4\},\{5\}\}$$

We now construct an equivalence relation $\sim$ on S using $\mathcal{A}$.

Examples: $1\sim 2$ since $1\in\{1,2,3\}$ and $2\in\{1,2,3\}$

$4\sim 4$ since $4\in\{4\}$ and $4\in\{4\}$

$$\left.\begin{array}{l}\bar{1}=\{1,2,3\}=\bar{2}=\bar{3}\\ \bar{4}=\{4\}\\ \bar{5}=\{5\}\end{array}\right\}\text{ equivalence classes}$$

$$S/\sim\;=\{\bar{1},\bar{4},\bar{5}\}=\{\{1,2,3\},\{4\},\{5\}\}=\mathcal{A}$$

$$\sim\;=\{\underbrace{(1,1),(1,2),(1,3),(2,2),(2,1),(2,3),(3,1),(3,2),(3,3)}_{\text{comes from }\{1,2,3\}},\underbrace{(4,4)}_{\text{comes from }\{4\}},\underbrace{(5,5)}_{\text{comes from }5}\}$$

proof of theorem: Recall $a \sim b$ iff there exists $A \in \mathcal{A}$ with $a \in A$ and $b \in A$.

① (reflexive) Let $x \in S$.

By the def of partition, $S = \bigcup_{A \in \mathcal{A}} A$.

So, $x \in \bigcup_{A \in \mathcal{A}} A$.

So, there exists $A \in \mathcal{A}$ with $x \in A$.

Thus, $x \sim x$ [since $x \in A$ and $x \in A$].

(symmetric) Let $x, y \in S$ and $x \sim y$.

Then there exists $A \in \mathcal{A}$ with
$x \in A$ and $y \in A$.

So, $y \in A$ and $x \in A$.

Thus, $y \sim x$.

(transitive) Let $x, y, z \in S$ and $x \sim y$ and $y \sim z$.

Since $x \sim y$ there exists $A \in \mathcal{A}$ with $x \in A$ and $y \in A$.

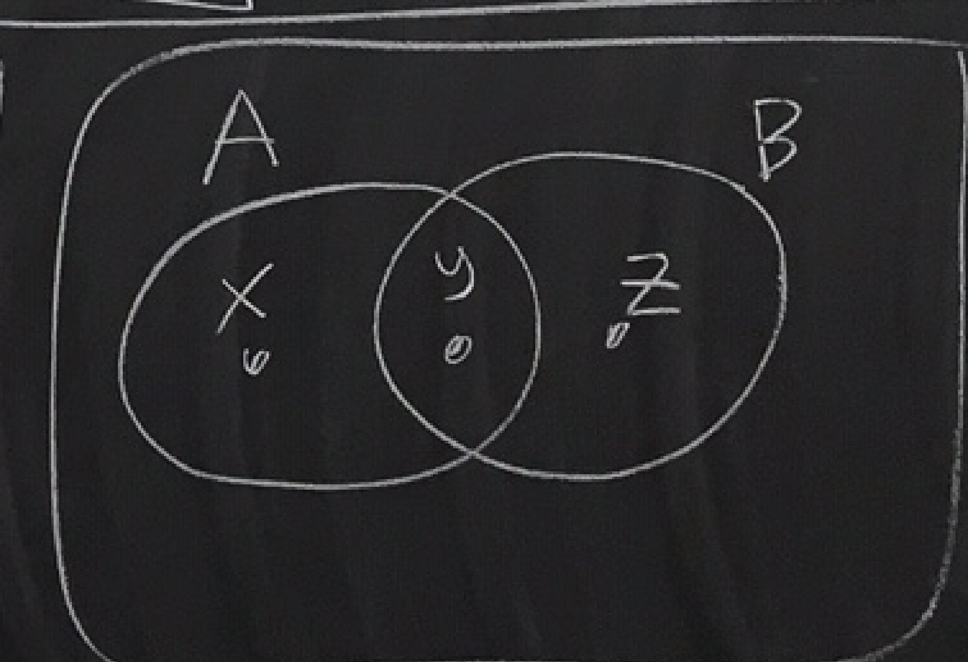
Since $y \sim z$ there exists $B \in \mathcal{A}$ with $y \in B$ and $z \in B$.

Since $y \in A \cap B$ we know $A \cap B \neq \phi$.

Since $\mathcal{A}$ is a partition we must have $A = B$ [since if $A \neq B$, the partition def part 3 would imply $A \cap B = \phi$].

Thus, $x \in A$ and $z \in A$.

So, $x \sim z$ $\boxed{1}$



S

② We want to show that $S/{\sim} = \mathcal{A}$.

$\boxed{\subseteq}$: Let $\bar{a} \in S/{\sim}$.

Pick the unique $A \in \mathcal{A}$ with $a \in A$.

Then $\bar{a} = A$ by the def of $\sim$.

So, $\bar{a} \in \mathcal{A}$.

$\boxed{\supseteq}$: Let $A \in \mathcal{A}$.

Pick any $a \in A$.

Then by the def of $\sim$ we have $\bar{a} = A$.

So, $A = \bar{a} \in S/{\sim}$.