Monday
10/14

Currently

Test 2 – Weds Nov 6

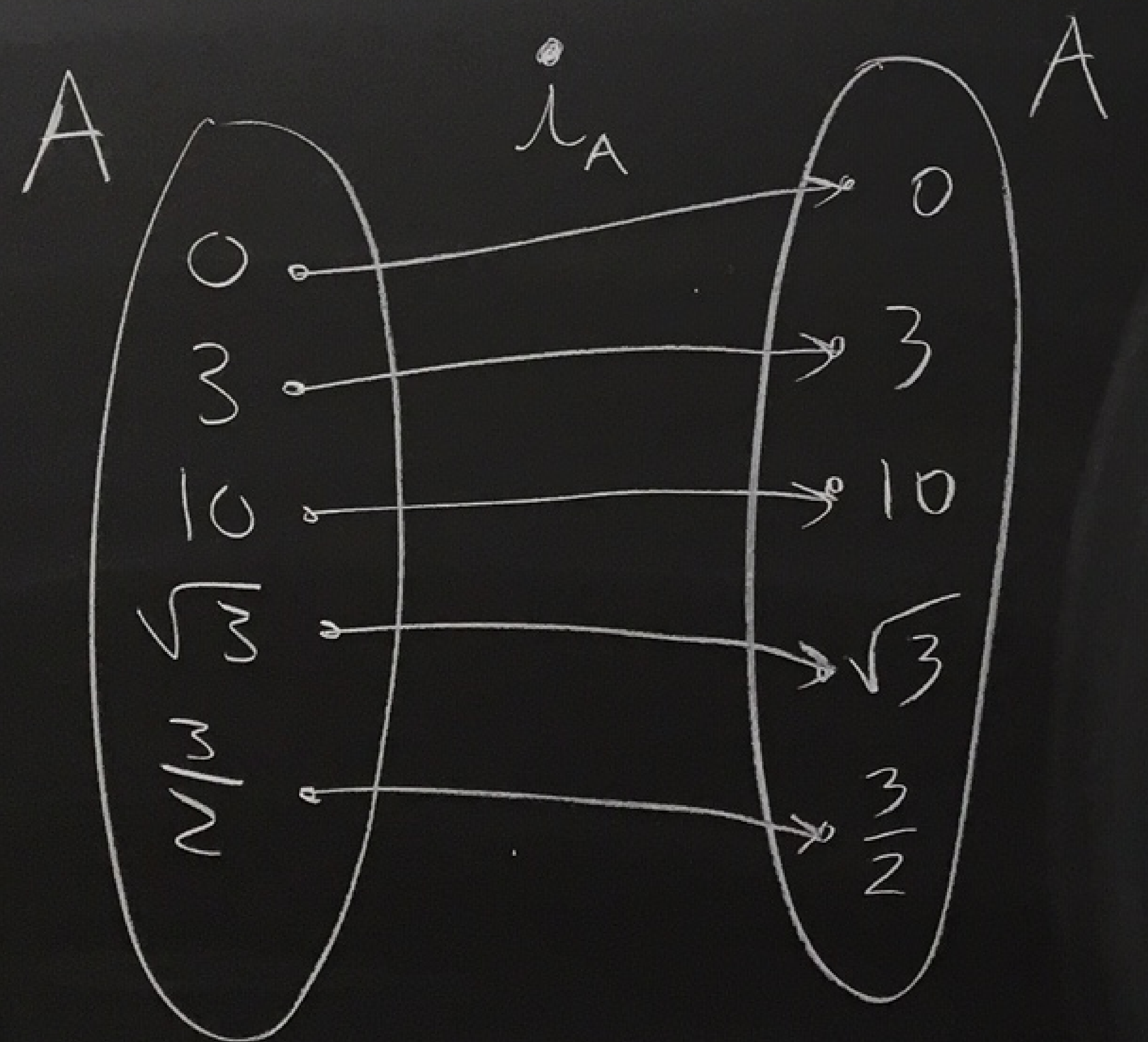
Move test 2 to Weds – Nov 13

Functions continued...

Ex: Let A be any set.
The identity function on A is the function $i_A : A \to A$ where $i_A(x) = x$ for all $x \in A$.

Sometimes we just use i instead of i_A.

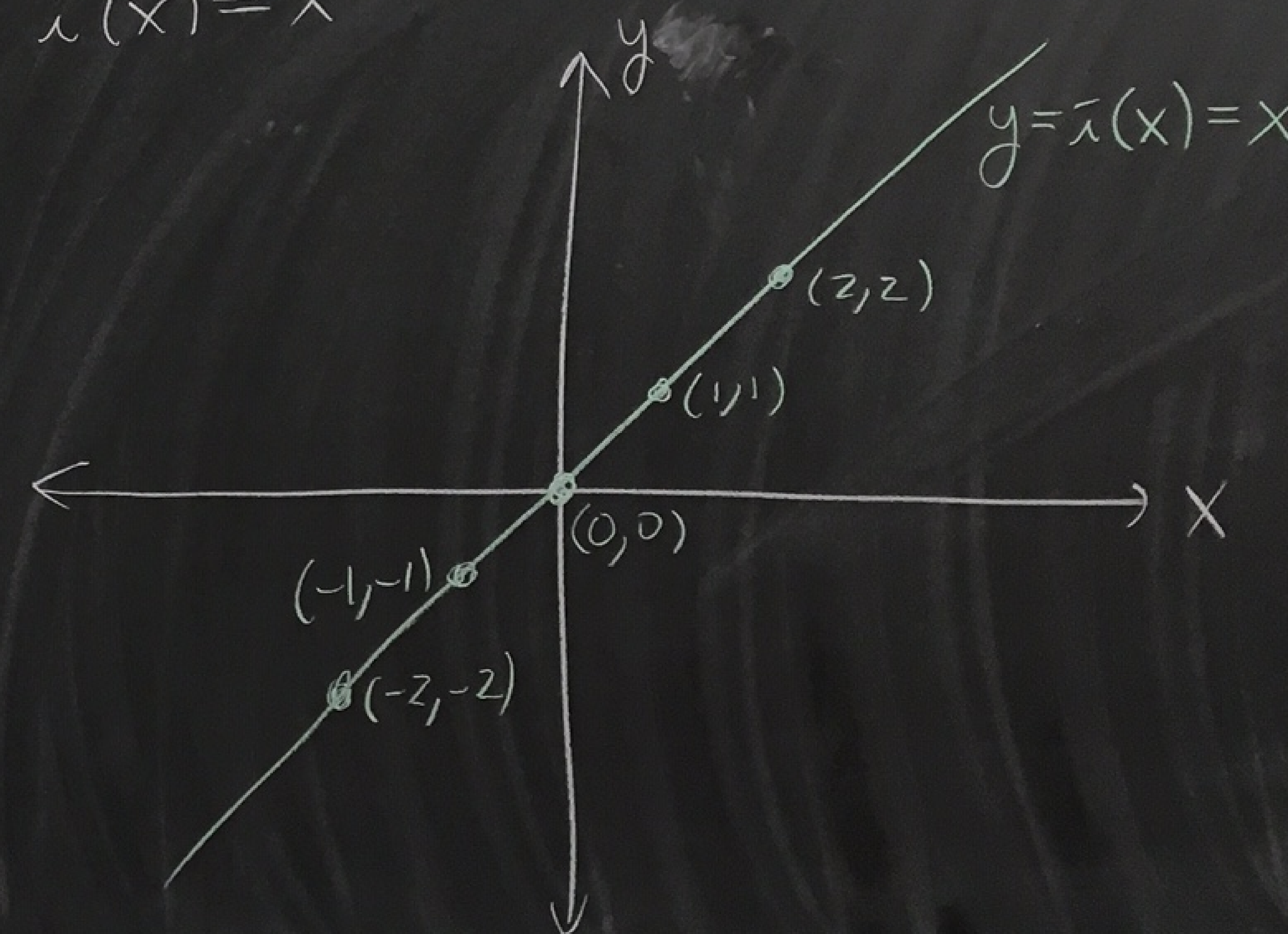
Ex: $A = \{0, 3, 10, \sqrt{3}, \frac{3}{2}\}$



Ex: $A = \mathbb{R}$

$i : \mathbb{R} \to \mathbb{R} \longleftarrow \boxed{i = i_{\mathbb{R}}}$

$i(x) = x$



Ex: Let $n \in \mathbb{Z}$, $n \geq 2$.

Define the reduction modulo n map

to be $\pi_n : \mathbb{Z} \longrightarrow \mathbb{Z}_n$

where $\pi_n(x) = \overline{x}$.

map means function

mapping is also used

Ex: $n=3$, $\pi_3 : \mathbb{Z} \to \mathbb{Z}_3$ where $\pi_3(x) = \overline{x}$

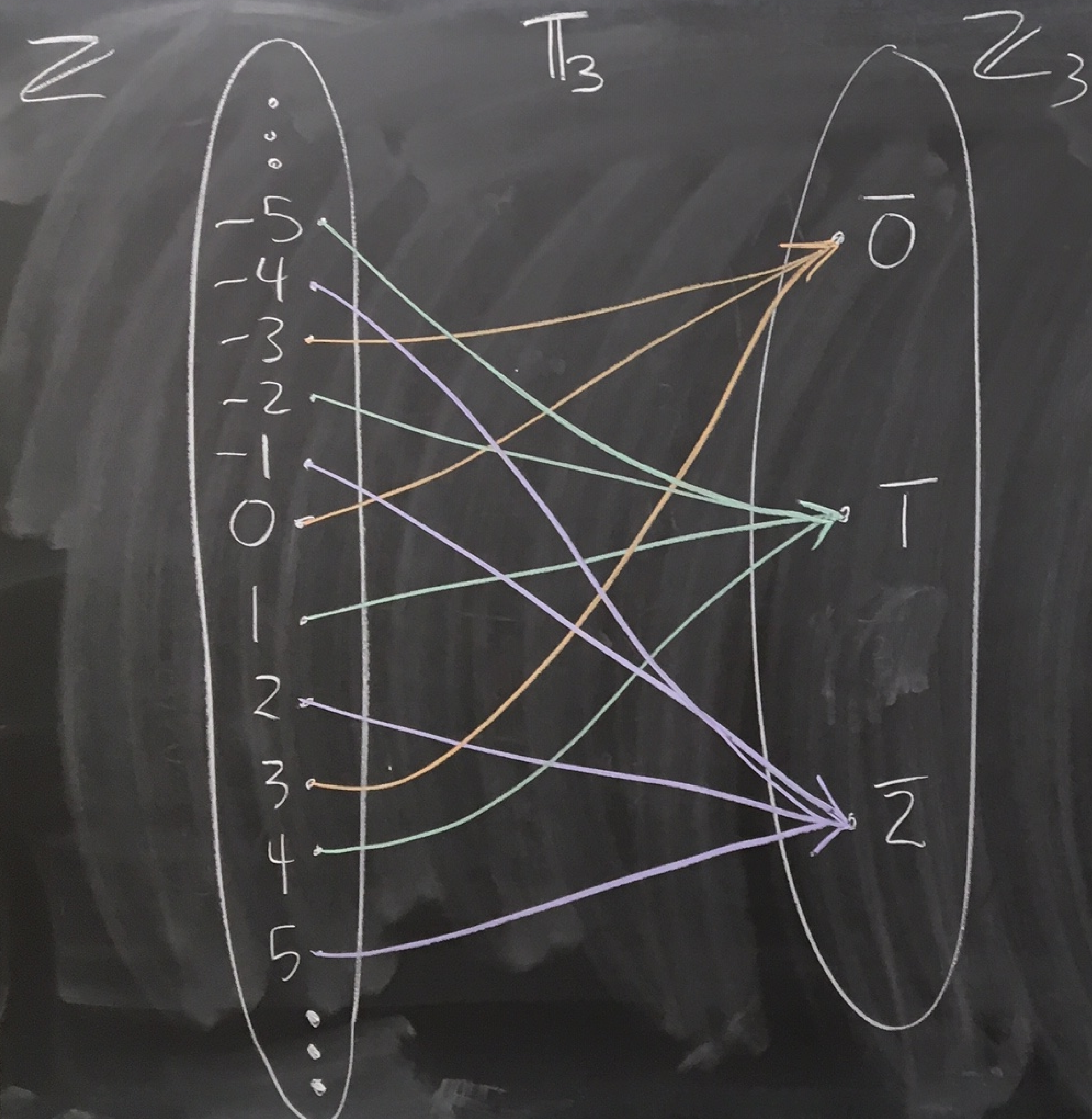
$\boxed{\mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}}$

$\pi_3(1) = \overline{1}$

$\pi_3(4) = \overline{4} = \overline{1}$

$\pi_3(-10) = \overline{-10} = \overline{2}$

$\boxed{-10-2 = -12 = -4 \cdot 3 \quad 3 \mid (-12)}$



$$\text{domain}(\pi_3) = \mathbb{Z}$$

$$\text{codomain}(\pi_3) = \mathbb{Z}_3$$

$$\text{range}(\pi_3) = \mathbb{Z}_3$$

Ex: Suppose you and your friend Peter want to define a function.

You say "what about the function $f: \mathbb{Q} \to \mathbb{Q}$ given by $f\left(\frac{a}{b}\right) = \frac{b}{a}$."

$\mathbb{Q} \leftarrow$ rational numbers ie fractions

Peter says: "I don't know if that function is ok. What about $f\left(\frac{0}{3}\right) = \frac{3}{0}$? Isn't that undefined?"

You say "Oh, you're right. Good call."

Ex: Then you say, "ok, I've got it.
what about $g: \mathbb{Q} \to \mathbb{Q}$ where $g\left(\frac{a}{b}\right) = a$?
That totally works. For example,
$g\left(\frac{7}{11}\right) = 7$ and $g\left(\frac{0}{3}\right) = 0$.
See everything's ok."
Peter says: " Hey, $g\left(\frac{14}{22}\right) = 14$
and $g\left(\frac{7}{11}\right) = 7$ but $\frac{14}{22} = \frac{7}{11}$!!!
That still is no good."
You say, "Ah man, too bad."

How to show that $f: A \to B$ is well-defined

Check the following:

1. If $a \in A$, then $f(a) \in B$.
2. If some or all of the elements in A can be expressed in more than one way then we must check that if $a_1, a_2 \in A$ and $a_1 = a_2$ then $f(a_1) = f(a_2)$.

Ex: Is $f: \mathbb{Q} \to \mathbb{Q}$

given by $f\left(\frac{a}{b}\right) = \left(\frac{a}{b}\right)^2$

Well-defined?

Recall

$\frac{a}{b} = \frac{c}{d}$ in $\mathbb{Q}$

iff

$ad = bc$

Yes. f is well-defined

① Let $\frac{a}{b} \in \mathbb{Q}$. So, $a, b \in \mathbb{Z}$, $b \neq 0$.

Then $f\left(\frac{a}{b}\right) = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$.

And, $a^2 \in \mathbb{Z}$, $b^2 \in \mathbb{Z}$, and $b^2 \neq 0$ (since $b \neq 0$).

So, $\frac{a^2}{b^2} \in \mathbb{Q}$.

② Let $\frac{a}{b}, \frac{c}{d} \in \mathbb{Q}$ where $\frac{a}{b} = \frac{c}{d}$.

We need to show that $f\left(\frac{a}{b}\right) = f\left(\frac{c}{d}\right)$.

Since $\frac{a}{b} = \frac{c}{d}$ we have that $ad = bc$.

Thus, $(ad)^2 = (bc)^2$.

So, $a^2 d^2 = b^2 c^2$.

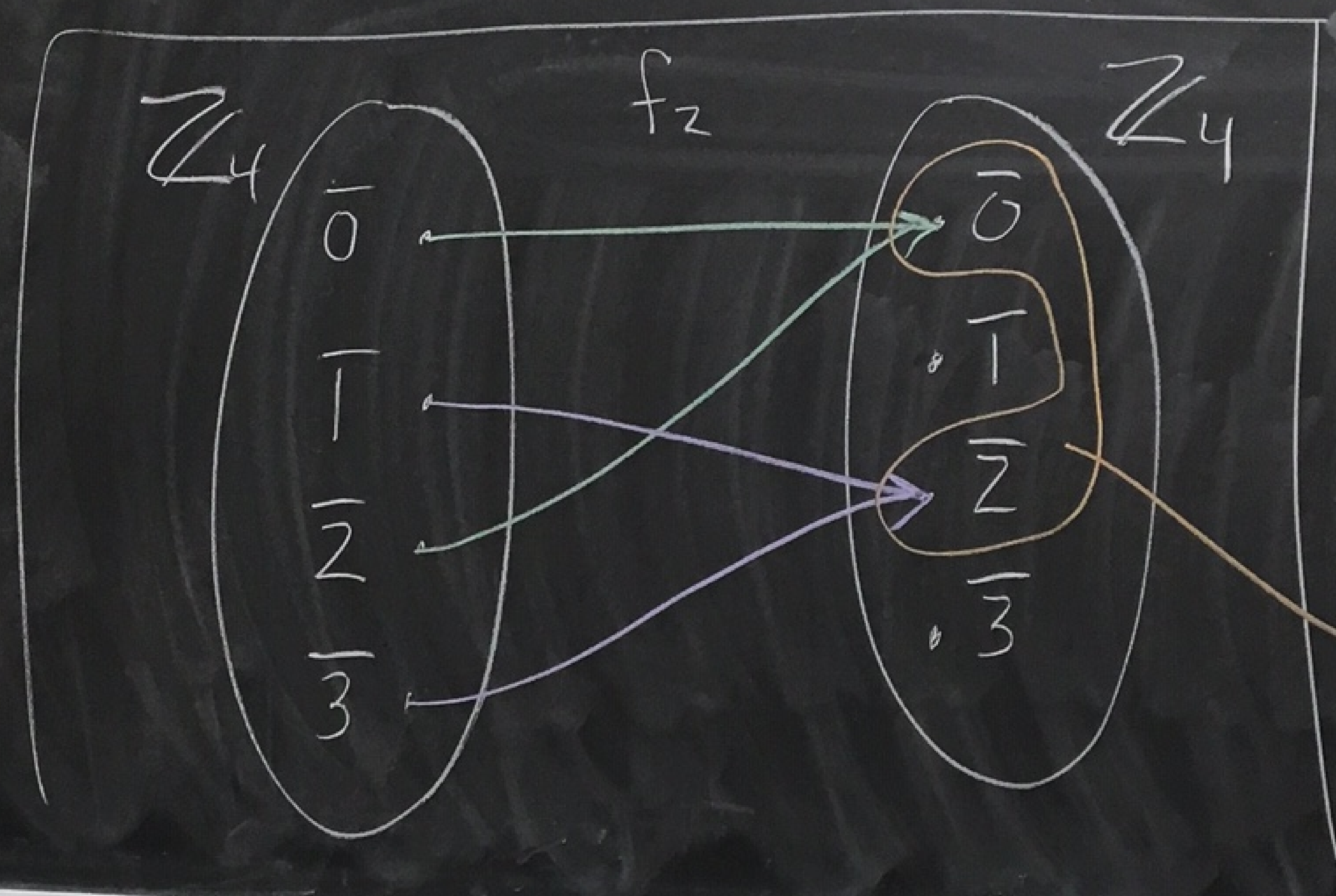
Hence, $\frac{a^2}{b^2} = \frac{c^2}{d^2}$.

So, $f\left(\frac{a}{b}\right) = f\left(\frac{c}{d}\right)$.

$\blacksquare$

Ex: Let $n \in \mathbb{Z}$, $n \geq 2$.
Pick $a \in \mathbb{Z}$.
Define
$$f_a : \mathbb{Z}_n \to \mathbb{Z}_n$$
by
$$f_a(\overline{x}) = \overline{a} \cdot \overline{x}$$

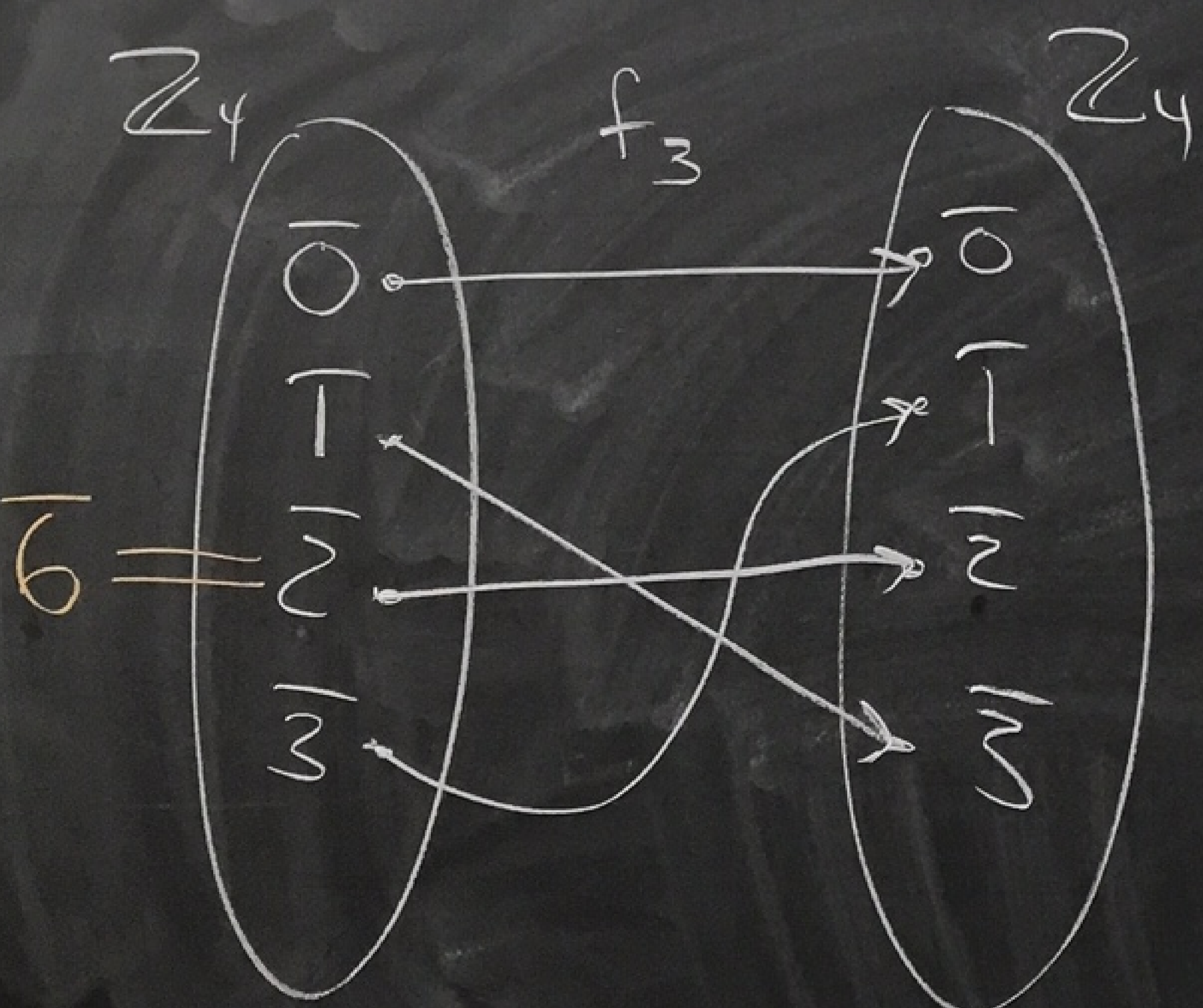
For example,
let $n = 4$ and $a = 2$.
$f_2 : \mathbb{Z}_4 \to \mathbb{Z}_4$ where $f_2(\overline{x}) = \overline{2} \cdot \overline{x}$



$f_2(\overline{0}) = \overline{2} \cdot \overline{0} = \overline{0}$
$f_2(\overline{1}) = \overline{2} \cdot \overline{1} = \overline{2}$
$f_2(\overline{2}) = \overline{2} \cdot \overline{2} = \overline{4} = \overline{0}$
$f_2(\overline{3}) = \overline{2} \cdot \overline{3} = \overline{6} = \overline{2}$
$\text{range}(f_2) = \{\overline{0}, \overline{2}\}$

$f_3 : \mathbb{Z} \to \mathbb{Z}_4$

$f_3(\bar{x}) = \bar{3}\cdot\bar{x}$



$f_3(\bar{0}) = \bar{3}\cdot\bar{0} = \bar{0}$

$f_3(\bar{1}) = \bar{3}\cdot\bar{1} = \bar{3}$

$f_3(\bar{2}) = \bar{3}\cdot\bar{2} = \bar{6} = \bar{2}$

$\boxed{f_3(\bar{6}) = \bar{3}\cdot\bar{6} = \overline{18} = \bar{2}}$

$f_3(\bar{3}) = \bar{3}\cdot\bar{3} = \bar{9} = \bar{1}$

$\text{range}(f_3) = \mathbb{Z}_4$

Proposition: Let $n, a \in \mathbb{Z}$ with $n \geq 2$.
Then $f_a : \mathbb{Z}_n \to \mathbb{Z}_n$ given by $f_a(\overline{x}) = \overline{a} \cdot \overline{x}$ is well-defined.

proof:

① Let $\overline{x} \in \mathbb{Z}_n$ where $x \in \mathbb{Z}$.

Since $a \in \mathbb{Z}$, we know $\overline{a} \in \mathbb{Z}_n$.
And $ax \in \mathbb{Z}$ so $\overline{ax} \in \mathbb{Z}_n$.
Thus, $f_a(\overline{x}) = \overline{a} \cdot \overline{x} = \overline{ax} \in \mathbb{Z}_n$.

② Suppose $x, y \in \mathbb{Z}$ and $\overline{x} = \overline{y}$ in $\mathbb{Z}_n$. Is $f_a(\overline{x}) = f_a(\overline{y})$?

Since $\overline{a} = \overline{a}$ and $\overline{x} = \overline{y}$ we know $\overline{a} \cdot \overline{x} = \overline{a} \cdot \overline{y}$ (from an earlier theorem in class.)
So, $f_a(\overline{x}) = f_a(\overline{y})$. $\blacksquare$