

3/9
Monday
Week 8

Def 71

Given a positive integer n define

$$I(n) = \frac{\sigma(n)}{n}$$

$I(n)$ is called the abundancy index/ratio

of n .

Ex 72

$$I(4) = \frac{\sigma(4)}{4} = \frac{1+2+4}{4} = \frac{7}{4} < 2$$

$$I(6) = \frac{\sigma(6)}{6} = \frac{1+2+3+6}{6} = \frac{12}{6} = 2$$

Fact 73 Let n be a positive integer.

Then, n is perfect

$$\text{iff } \sigma(n) = 2n \quad \text{iff } I(n) = \frac{\sigma(n)}{n} = 2.$$

Fact 74

Let n be a positive integer where $n \geq 2$.

Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$

be the prime factorization of n , where

p_1, p_2, \dots, p_k are
distinct primes.

Then

$$I(n) = I(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}) = \frac{\sigma(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k})}{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}}$$

$$= \frac{\sigma(p_1^{\alpha_1}) \sigma(p_2^{\alpha_2}) \dots \sigma(p_k^{\alpha_k})}{p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}} = \frac{\sigma(p_1^{\alpha_1})}{p_1^{\alpha_1}} \cdot \frac{\sigma(p_2^{\alpha_2})}{p_2^{\alpha_2}} \dots \frac{\sigma(p_k^{\alpha_k})}{p_k^{\alpha_k}}$$

$$= I(p_1^{\alpha_1}) I(p_2^{\alpha_2}) \dots I(p_k^{\alpha_k})$$

(Thm 35)

Fact 75 Let p be prime and α be a positive integer.

Then

$$\begin{aligned} I(p^\alpha) &= \frac{\sigma(p^\alpha)}{p^\alpha} = \frac{1 + p + p^2 + \dots + p^\alpha}{p^\alpha} \\ &= 1 + \frac{1}{p} + \frac{1}{p^2} + \dots + \frac{1}{p^\alpha} \end{aligned}$$

Def 76

Let n be a positive integer.

Define $w(n)$ to be the number of distinct primes in the prime factorization of n .

Ex 77

$$\omega(125) = \omega(5^3) = 1$$

$$\omega(12) = \omega(2^2 \cdot 3^1) = 2$$

$$\omega(7^3 \cdot 11^{105} \cdot 13^2 \cdot 17^1) = 4$$

We are going
to show that

$$\omega(n) \geq 4$$

if n is an
odd perfect number.

Thm 78

Let n be an odd perfect number.

Then $w(n) \geq 2$.

Proof: Suppose n is an odd perfect number and $w(n)=1$.

Then $n = p^\alpha$ where p is an odd prime and α is a positive integer.

Since n is perfect, $I(n) = 2$.

So,

$$2 = I(n)$$

$$= I(p^\alpha)$$

$$= 1 + \frac{1}{p} + \frac{1}{p^2} + \dots + \frac{1}{p^\alpha}$$

M13

$$< 1 + \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} + \dots$$

$$\sum_{k=0}^{\infty} \left(\frac{1}{p}\right)^k$$

Geometric series

$$1+r+r^2+r^3+\dots = \frac{1}{1-r}$$

when $-1 < r < 1$

$$\frac{1}{1-\frac{1}{P}} = \frac{P}{P-1} \leq \frac{3}{2} = 1.5.$$

$P \geq 3$ & lemma 79

This is a contradiction
since $2 < 1.5$ is nonsense.
Thus, $w(n) \geq 2$. \square

Lemma 79 If $P \geq c > 1$,

$$\text{then } \frac{P}{P-1} \leq \frac{c}{c-1}$$

pf of lemma:

$$\begin{aligned} P \geq c > 1 &\Rightarrow -P \leq -c \\ &\Rightarrow PC - P \leq PC - c \\ &\Rightarrow P(c-1) \leq c(P-1) \\ &\Rightarrow \frac{P}{P-1} \leq \frac{c}{c-1} \end{aligned}$$



Thm 80

Let n be an odd perfect number.

Then $w(n) \geq 3$.

Proof: Suppose n is an odd perfect number. We know from Thm 78 that $w(n) \geq 2$. Let's rule out $w(n) = 2$.

Suppose $n = p_1^{\alpha_1} p_2^{\alpha_2}$

where p_1, p_2 are odd primes
and α_1, α_2 are positive integers.

We know $p_1 \geq 3$ and $p_2 \geq 5$.

Since n is perfect

$$\text{we know } I(n) = \frac{\sigma(n)}{n} = 2$$

M13

$$\begin{aligned} & \text{geometric series} \\ & \frac{1}{1+r+r^2+r^3+\dots} \\ & = \frac{1}{1-r} \\ & \text{if } -1 < r \end{aligned}$$

We get that

$$2 = I(n) = I(P_1^{\alpha_1} P_2^{\alpha_2}) \stackrel{\text{Fact 74}}{=} I(P_1^{\alpha_1}) I(P_2^{\alpha_2}).$$

Fact 75

$$\stackrel{\text{Fact 75}}{=} \left(1 + \frac{1}{P_1} + \frac{1}{P_1^2} + \dots + \frac{1}{P_1^{\alpha_1}}\right) \left(1 + \frac{1}{P_2} + \frac{1}{P_2^2} + \dots + \frac{1}{P_2^{\alpha_2}}\right)$$

$$< \left(1 + \frac{1}{P_1} + \frac{1}{P_1^2} + \frac{1}{P_1^3} + \dots\right) \left(1 + \frac{1}{P_2} + \frac{1}{P_2^2} + \frac{1}{P_2^3} + \dots\right)$$

$$\stackrel{\text{Lemma 79}}{=} \left(\frac{1}{1 - \frac{1}{P_1}}\right) \cdot \left(\frac{1}{1 - \frac{1}{P_2}}\right) = \frac{P_1}{P_1 - 1} \cdot \frac{P_2}{P_2 - 1} \leq \frac{3}{2} \cdot \frac{5}{4} = \frac{15}{8} = 1.875$$

geometric sum

$$1 + r + r^2 + r^3 + \dots$$

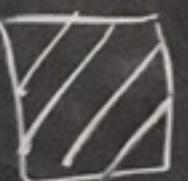
$$= \frac{1}{1 - r}$$

If $-1 < r < 1$

Fact 74

Contradiction.

$2 < 1.875$?

Nonsense! 

Theorem 81

Let n be an odd perfect number.

Then $\omega(n) \geq 4$.

Pf: Let n be an odd perfect number. We already showed $\omega(n) \geq 3$.

Let's rule out $\omega(n) = 3$.

Suppose $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$ is the prime factorization of n .

Since n is odd, the p_i are odd.

So we can assume

$$p_1 \geq 3, p_2 \geq 5, p_3 \geq 7.$$

We now show $3 \mid n$.

If $3 \nmid n$, then

$\Rightarrow p_1 \geq 5, p_2 \geq 7, p_3 \geq 11$.

Then, by the same ideas as

Thm 8.0 we would have

$$\begin{aligned} 2 &= I(n) = I(p_1^{\alpha_1}) I(p_2^{\alpha_2}) I(p_3^{\alpha_3}) \\ &< \left(1 + \frac{1}{p_1} + \frac{1}{p_1^2} + \dots\right) \left(1 + \frac{1}{p_2} + \frac{1}{p_2^2} + \dots\right) \left(1 + \frac{1}{p_3} + \frac{1}{p_3^2} + \dots\right) \\ &= \frac{p_1}{p_1-1} \cdot \frac{p_2}{p_2-1} \cdot \frac{p_3}{p_3-1} \leq \frac{5}{4} \cdot \frac{7}{6} \cdot \frac{11}{10} \\ &= \frac{77}{48} \approx 1.6041 \end{aligned}$$

Contradiction.

So we can assume $p_1 = 3$

We can also show $p_2 = 5$.

If $p_1 = 3, p_2 \geq 7, p_3 \geq 11$

then as before

$$\begin{aligned} 2 &= I(n) < \frac{p_1}{p_1-1} \cdot \frac{p_2}{p_2-1} \cdot \frac{p_3}{p_3-1} \\ &\leq \frac{3}{2} \cdot \frac{7}{6} \cdot \frac{11}{10} \\ &= \frac{231}{120} \approx 1.925 \end{aligned}$$

contradiction

So, $P_1 = 3$ and $P_2 = 5$ and $P_3 \geq 7$

Thus,

$$\begin{aligned} 2 = I(n) &\leq \frac{P_1}{P_1 - 1} \cdot \frac{P_2}{P_2 - 1} \cdot \frac{P_3}{P_3 - 1} \\ &\leq \frac{3}{2} \cdot \frac{5}{4} \cdot \frac{P_3}{P_3 - 1} \end{aligned}$$

$$\text{So, } \frac{16}{15} < \frac{P_3}{P_3 - 1} \cdot \text{ So, } 16P_3 - 16 < 15P_3.$$

Thus, $P_3 < 16$.

Summary: $n = 3^{\alpha_1} 5^{\alpha_2} p_3^{\alpha_3}$
where $p_3 = 7, 11$, or 13 .

Euler: If n is an odd perfect number
then $n = q^e p_1^{2e_1} p_2^{2e_2} \dots p_k^{2e_k}$
 $q \equiv e \equiv 1 \pmod{4}$. So, e is odd.

Thm
64

case 1: $n = 3^{\alpha_1} 5^{\alpha_2} 7^{\alpha_3}$

Since $3 \equiv 7 \equiv 3 \pmod{4}$

and $5 \equiv 1 \pmod{4}$,

by Euler $\alpha_1 \geq 2$ and $\alpha_3 \geq 2$.

So,

$$\begin{aligned} I(n) &= \left(1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^{\alpha_1}}\right) \left(1 + \frac{1}{5} + \frac{1}{5^2} + \dots + \frac{1}{5^{\alpha_2}}\right) \left(1 + \frac{1}{7} + \frac{1}{7^2} + \dots + \frac{1}{7^{\alpha_3}}\right) \\ &= \left(1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^2}\right) \left(1 + \frac{1}{5}\right) \left(1 + \frac{1}{7} + \frac{1}{7^2}\right) = \frac{494}{245} \approx 2.0163\dots \end{aligned}$$

Contradiction.

$$\text{case 2: } n = 3^{\alpha_1} 5^{\alpha_2} 11^{\alpha_3}$$

Since $3 \equiv 11 \equiv 3 \pmod{4}$ and $5 \equiv 1 \pmod{4}$

We get by Euler, $\alpha_2 \geq 1$, $\alpha_1 \geq 2$, $\alpha_3 \geq 2$.

$$\text{case 2A: } \alpha_2 = 1$$

Then,

$$\begin{aligned} Z &= I(n) < \left(1 + \frac{1}{3} + \frac{1}{3^2} + \dots\right) \left(1 + \frac{1}{5}\right) \left(1 + \frac{1}{11} + \frac{1}{11^2} + \dots\right) \\ &< \frac{3}{2} \cdot \frac{6}{5} \cdot \frac{11}{10} = \frac{99}{50} \end{aligned}$$

contradiction.

$$\text{case 2B: } \alpha_2 \geq 2$$

If $\alpha_1 = 2$, then

$$\begin{aligned} 2n &= \sigma(n) = \sigma(3^2) \sigma(5^{\alpha_2}) \sigma(11^{\alpha_3}) \\ &= (1+3+3^2) \sigma(5^{\alpha_2}) \sigma(11^{\alpha_3}) \\ &= 13 \sigma(5^{\alpha_2}) \sigma(11^{\alpha_3}) \end{aligned}$$

But $13 \nmid 2n$.

So this case can't happen.

So, $\alpha_1 \geq 4$, $\alpha_2 \geq 2$, $\alpha_3 \geq 2$.

Then

$$\begin{aligned} Z &= I(n) \\ &= \left(1 + \frac{1}{3} + \dots + \frac{1}{3^{\alpha_1}}\right) \left(1 + \frac{1}{5} + \dots + \frac{1}{5^{\alpha_2}}\right) \left(1 + \frac{1}{11} + \dots + \frac{1}{11^{\alpha_3}}\right) \\ &\geq \left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4}\right) \left(1 + \frac{1}{5} + \frac{1}{5^2}\right) \left(1 + \frac{1}{11} + \frac{1}{11^2}\right) = \frac{4123}{2025} \approx 2.036. \end{aligned}$$

Contradiction.

Case 3: $n = 3^{\alpha_1} 5^{\alpha_2} | 3^{\alpha_3}$

Case 3A: $\alpha_2 = 1$

Then

$$\begin{aligned} Z &= I(n) = \left(1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^{\alpha_1}}\right) \left(1 + \frac{1}{5}\right) \left(1 + \frac{1}{13} + \dots + \frac{1}{13^{\alpha_3}}\right) \\ &< \frac{3}{2} \cdot \left(1 + \frac{1}{5}\right) \cdot \frac{13}{12} = \frac{39}{20} \quad \text{contradiction} \end{aligned}$$

case 3B: $\alpha_2 \geq 2$

Since $3 \not\equiv 1 \pmod{4}$, Euler says that α_1 is even.

case 3B-1 : $\alpha_1 = 2$

$$2 = I(n) = \left(1 + \frac{1}{3} + \frac{1}{3^2}\right) \left(1 + \frac{1}{5} + \dots + \frac{1}{5^{\alpha_2}}\right) \left(1 + \frac{1}{13} + \dots + \frac{1}{13^{\alpha_3}}\right)$$

$$< \left(1 + \frac{1}{3} + \frac{1}{3^2}\right) \cdot \frac{5}{4} \cdot \frac{13}{12} = \frac{845}{432} \approx 1,956$$

Can't happen

case 3B-2 : $\alpha_1 \geq 4$

If $\alpha_3 = 1$, then $2n = \sigma(n) = \sigma(3^{\alpha_1})\sigma(5^{\alpha_2})\sigma(13)$
 $= 14 \sigma(3^{\alpha_1})\sigma(5^{\alpha_2})$

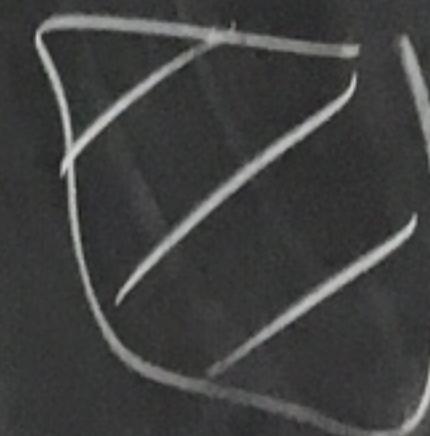
Then $14 \mid 2n$. Or $7 \mid n$. Can't happen.

So, $\alpha_3 \geq 2$.

Thus,

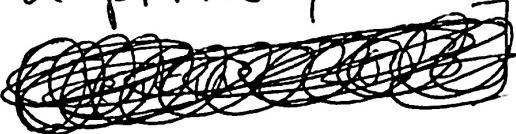
$$Z = I(n) \geq \left(1 + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \frac{1}{3^4}\right) \left(1 + \frac{1}{5} + \frac{1}{5^2}\right) \left(1 + \frac{1}{13} + \frac{1}{13^2}\right)$$
$$= \frac{228811}{114075} \approx 2.00579\dots$$

Contradiction,



If n is an odd perfect number, then

① n is divisible by a prime P
with $P > 10^8$



[Goto & Ohno, 2008]

② $n > 10^{1500}$ [Ochem & Rao, 2012]

③ $105 \nmid n$, [$105 = 3 \cdot 5 \cdot 7$]

④ $n \equiv 1 \pmod{12}$ or $n \equiv 117 \pmod{468}$
or $n \equiv 81 \pmod{324}$

⑤ $n = q^\alpha p_1^{2e_1} p_2^{2e_2} \dots p_k^{2e_k}$

where q, p_1, p_2, \dots, p_k are distinct
odd primes, α, e_1, \dots, e_k are positive integers
and $\alpha \equiv q \equiv 1 \pmod{4}$ [Euler]

⑥ n has at least 10 distinct
prime factors [Nielsen, 2015]