Weds
2/5
week 3

Euclid proved a nice fact about perfect numbers.

Motivation:

1st perfect # $\longrightarrow 6 = 2 \cdot \underbrace{3}_{\text{prime}} = 2^1 \cdot \underbrace{(2^2 - 1)}_{\text{prime}}$

2nd perfect # $\longrightarrow 28 = 2^2 \cdot \underbrace{7}_{\text{prime}} = 2^2 \underbrace{(2^3 - 1)}_{\text{prime}}$

3rd perfect # $\longrightarrow$ $496 = 2^4 \cdot \underbrace{31}_{\text{prime}} = 2^4 \underbrace{(2^5 - 1)}_{\text{prime}}$

4th perfect # $\longrightarrow$ $8128 = 2^6 \cdot \underbrace{127}_{\text{prime}} = 2^6 \cdot \underbrace{(2^7 - 1)}_{\text{prime}}$

Math 4900 class conjecture:

Even perfect numbers look like $2^{n-1}(2^n - 1)$ where n is prime and $2^n - 1$ is prime.

Note: You can't just have n be prime. Check out this example.

Try $n = 11$.

$$2^{10}(2^{11} - 1) = 2^{10} \cdot \underbrace{2047}_{\text{not prime}}$$

$$= 2^{10} \cdot 23 \cdot 89$$

$\underbrace{1}_{\text{rime}} = 2^4\underbrace{(2^5-1)}_{\text{prime}}$

$\underbrace{7}_{\text{me}} = 2^6 \cdot \underbrace{(2^7-1)}_{\text{prime}}$

<u>Note:</u> You can't just have n be prime.
Check out this example.

Try $n=11$.

$$2^{10}(2^{11}-1) = 2^{10} \cdot \underbrace{2047}_{\text{not prime}}$$

$$= 2^{10} \cdot 23 \cdot 89$$

If you test if

$$2{,}096{,}128 = 2^{10}(2^{11}-1)$$

is perfect, you'll find its not perfect.

List of Known Mersenne Prime Numbers

#	2^P-1	Digits	Date Discovered	Discovered By	Method / Hardware	Perfect Number
1	2^2-1	1	c. 500 BCE	Ancient Greek mathematicians		$2^1 \cdot (2^2-1)$
2	2^3-1	1	c. 500 BCE	Ancient Greek mathematicians		$2^2 \cdot (2^3-1)$
3	2^5-1	2	c. 275 BCE	Ancient Greek mathematicians		$2^4 \cdot (2^5-1)$
4	2^7-1	3	c. 275 BCE	Ancient Greek mathematicians		$2^6 \cdot (2^7-1)$
5	$2^{13}-1$	4	1456	Anonymous	trial division	$2^{12} \cdot (2^{13}-1)$
6	$2^{17}-1$	6	1588	Pietro Cataldi	trial division	$2^{16} \cdot (2^{17}-1)$
7	$2^{19}-1$	6	1588	Pietro Cataldi	trial division	$2^{18} \cdot (2^{19}-1)$
8	$2^{31}-1$	10	1772	Leonhard Euler	Enhanced trial division	$2^{30} \cdot (2^{31}-1)$
9	$2^{61}-1$	19	1883	Ivan Mikheevich Pervushin	Lucas sequences	$2^{60} \cdot (2^{61}-1)$
10	$2^{89}-1$	27	1911 Jun	R. E. Powers	Lucas sequences	$2^{88} \cdot (2^{89}-1)$
11	$2^{107}-1$	33	1914 Jun 11	R. E. Powers	Lucas sequences	$2^{106} \cdot (2^{107}-1)$
12	$2^{127}-1$	39	1876 Jan 10	Édouard Lucas	Lucas sequences	$2^{126} \cdot (2^{127}-1)$
13	$2^{521}-1$	157	1952 Jan 30	Raphael M. Robinson	L-L / SWAC	$2^{520} \cdot (2^{521}-1)$
14	$2^{607}-1$	183	1952 Jan 30	Raphael M. Robinson	L-L / SWAC	$2^{606} \cdot (2^{607}-1)$
15	$2^{1,279}-1$	386	1952 Jun 25	Raphael M. Robinson	L-L / SWAC	$2^{1,278} \cdot (2^{1,279}-1)$
16	$2^{2,203}-1$	664	1952 Oct 07	Raphael M. Robinson	L-L / SWAC	$2^{2,202} \cdot (2^{2,203}-1)$
17	$2^{2,281}-1$	687	1952 Oct 09	Raphael M. Robinson	L-L / SWAC	$2^{2,280} \cdot (2^{2,281}-1)$
18	$2^{3,217}-1$	969	1957 Sep 08	Hans Riesel	L-L / BESK	$2^{3,216} \cdot (2^{3,217}-1)$
19	$2^{4,253}-1$	1,281	1961 Nov 03	Alexander Hurwitz	L-L / IBM 7090	$2^{4,252} \cdot (2^{4,253}-1)$

Marin Mersenne - Wiki | List of known Mers | List of perfect numbers | MATH 4900 - Senior Se

https://www.mersenne.org/primes/

19	$2^{4,253}-1$	1,281	1961 Nov 03	Alexander Hurwitz	L-L / IBM 7090	$2^{4,252} \cdot (2^{4,253}-1)$
20	$2^{4,423}-1$	1,332	1961 Nov 03	Alexander Hurwitz	L-L / IBM 7090	$2^{4,422} \cdot (2^{4,423}-1)$
21	$2^{9,689}-1$	2,917	1963 May 11	Donald B. Gillies	L-L / ILLIAC II	$2^{9,688} \cdot (2^{9,689}-1)$
22	$2^{9,941}-1$	2,993	1963 May 16	Donald B. Gillies	L-L / ILLIAC II	$2^{9,940} \cdot (2^{9,941}-1)$
23	$2^{11,213}-1$	3,376	1963 Jun 02	Donald B. Gillies	L-L / ILLIAC II	$2^{11,212} \cdot (2^{11,213}-1)$
24	$2^{19,937}-1$	6,002	1971 Mar 04	Bryant Tuckerman	L-L / IBM 360/91	$2^{19,936} \cdot (2^{19,937}-1)$
25	$2^{21,701}-1$	6,533	1978 Oct 30	Landon Curt Noll & Laura Nickel	L-L / CDC Cyber 174	$2^{21,700} \cdot (2^{21,701}-1)$
26	$2^{23,209}-1$	6,987	1979 Feb 09	Landon Curt Noll	L-L / CDC Cyber 174	$2^{23,208} \cdot (2^{23,209}-1)$
27	$2^{44,497}-1$	13,395	1979 Apr 08	Harry Lewis Nelson & David Slowinski	L-L / Cray 1	$2^{44,496} \cdot (2^{44,497}-1)$
28	$2^{86,243}-1$	25,962	1982 Sep 25	David Slowinski	L-L / Cray 1	$2^{86,242} \cdot (2^{86,243}-1)$
29	$2^{110,503}-1$	33,265	1988 Jan 28	Walter Colquitt & Luke Welsh	L-L / NEC SX-2	$2^{110,502} \cdot (2^{110,503}-1)$
30	$2^{132,049}-1$	39,751	1983 Sep 19	David Slowinski	L-L / Cray X-MP	$2^{132,048} \cdot (2^{132,049}-1)$
31	$2^{216,091}-1$	65,050	1985 Sep 01	David Slowinski	L-L / Cray X-MP/24	$2^{216,090} \cdot (2^{216,091}-1)$
32	$2^{756,839}-1$	227,832	1992 Feb 19	David Slowinski & Paul Gage	L-L / Maple on Harwell Lab Cray-2	$2^{756,838} \cdot (2^{756,839}-1)$
33	$2^{859,433}-1$	258,716	1994 Jan 04	David Slowinski & Paul Gage	L-L / Cray C90	$2^{859,432} \cdot (2^{859,433}-1)$
34	$2^{1,257,787}-1$	378,632	1996 Sep 03	David Slowinski & Paul Gage	L-L / Cray T94	$2^{1,257,786} \cdot (2^{1,257,787}-1)$
35	$2^{1,398,269}-1$	420,921	1996 Nov 13	GIMPS / Joel Armengaud	L-L / Prime95 on 90 MHz Pentium PC	$2^{1,398,268} \cdot (2^{1,398,269}-1)$
36	$2^{2,976,221}-1$	895,932	1997 Aug 24	GIMPS / Gordon Spence	L-L / Prime95 on 100 MHz Pentium PC	$2^{2,976,220} \cdot (2^{2,976,221}-1)$
37	$2^{3,021,377}-1$	909,526	1998 Jan 27	GIMPS / Roland Clarkson	L-L / Prime95 on 200 MHz Pentium PC	$2^{3,021,376} \cdot (2^{3,021,377}-1)$
38	$2^{6,972,593}-1$	2,098,960	1999 Jun 01	GIMPS / Nayan Hajratwala	L-L / Prime95 on 350 MHz Pentium II IBM Aptiva	$2^{6,972,592} \cdot (2^{6,972,593}-1)$
39	$2^{13,466,917}-1$	4,053,946	2001 Nov 14	GIMPS / Michael Cameron	L-L / Prime95 on 800 MHz Athlon Thunderbird	$2^{13,466,916} \cdot (2^{13,466,917}-1)$

6:26 PM
2/5/2020

Marin Mersenne - Wiki | List of known Mers | List of perfect numbers | MATH 4900 - Senior Se

https://www.mersenne.org/primes/

39	$2^{13,466,917}-1$	4,053,946	2001 Nov 14	GIMPS / Michael Cameron	L-L / Prime95 on 800 MHz Athlon Thunderbird	$2^{13,466,916} \cdot (2^{13,466,917}-1)$
40	$2^{20,996,011}-1$	6,320,430	2003 Nov 17	GIMPS / Michael Shafer	L-L / Prime95 on 2 GHz Dell Dimension	$2^{20,996,010} \cdot (2^{20,996,011}-1)$
41	$2^{24,036,583}-1$	7,235,733	2004 May 15	GIMPS / Josh Findley	L-L / Prime95 on 2.4 GHz Pentium 4 PC	$2^{24,036,582} \cdot (2^{24,036,583}-1)$
42	$2^{25,964,951}-1$	7,816,230	2005 Feb 18	GIMPS / Martin Nowak	L-L / Prime95 on 2.4 GHz Pentium 4 PC	$2^{25,964,950} \cdot (2^{25,964,951}-1)$
43	$2^{30,402,457}-1$	9,152,052	2005 Dec 15	GIMPS / Curtis Cooper & Steven Boone	L-L / Prime95 on 2 GHz Pentium 4 PC	$2^{30,402,456} \cdot (2^{30,402,457}-1)$
44	$2^{32,582,657}-1$	9,808,358	2006 Sep 04	GIMPS / Curtis Cooper & Steven Boone	L-L / Prime95 on 3 GHz Pentium 4 PC	$2^{32,582,656} \cdot (2^{32,582,657}-1)$
45	$2^{37,156,667}-1$	11,185,272	2008 Sep 06	GIMPS / Hans-Michael Elvenich	L-L / Prime95 on 2.83 GHz Core 2 Duo PC	$2^{37,156,666} \cdot (2^{37,156,667}-1)$
46	$2^{42,643,801}-1$	12,837,064	2009 Jun 04	GIMPS / Odd M. Strindmo	L-L / Prime95 on 3 GHz Core 2 PC	$2^{42,643,800} \cdot (2^{42,643,801}-1)$
47	$2^{43,112,609}-1$	12,978,189	2008 Aug 23	GIMPS / Edson Smith	L-L / Prime95 on Dell Optiplex 745	$2^{43,112,608} \cdot (2^{43,112,609}-1)$
48*	$2^{57,885,161}-1$	17,425,170	2013 Jan 25	GIMPS / Curtis Cooper	L-L / Prime95 on Intel Core2 Duo E8400 @ 3.00GHz	$2^{57,885,160} \cdot (2^{57,885,161}-1)$
49*	$2^{74,207,281}-1$	22,338,618	2016 Jan 07	GIMPS / Curtis Cooper	L-L / Prime95 on Intel i7-4790 @ 3.60GHz	$2^{74,207,280} \cdot (2^{74,207,281}-1)$
50*	$2^{77,232,917}-1$	23,249,425	2017 Dec 26	GIMPS / Jon Pace	L-L / Prime95 on Intel i5-6600 @ 3.30GHz	$2^{77,232,916} \cdot (2^{77,232,917}-1)$
51*	$2^{82,589,933}-1$	24,862,048	2018 Dec 07	GIMPS / Patrick Laroche	L-L / Prime95 on Intel i5-4590T @ 2.0GHz	$2^{82,589,932} \cdot (2^{82,589,933}-1)$

* Provisional ranking, not all candidates between M43,112,609 and M82,589,933 have been eliminated

6:28 PM
2/5/2020

$\boxed{\text{Thm 29 (Euclid)}}$

$\boxed{\text{If } 2^n-1 \text{ is prime then } 2^{n-1}(2^n-1) \text{ is a perfect number.}}$

<u>Proof:</u> Let $x = 2^{n-1}(2^n-1)$.

Note that $n \geq 2$ since 2^n-1 is prime.

We need to sum the divisors of x.

Since 2^n-1 is prime we have the prime factorization of x.

It is

$$x = 2^{n-1} \cdot (2^n-1)^1$$

The divisors of x are:

$$2^0\cdot(2^n-1)^0,\ 2^1(2^n-1)^0,\ 2^2(2^n-1)^0,\ \dots,\ 2^{n-1}(2^n-1)^0 \longleftarrow \text{1st row}$$

$$2^0\cdot(2^n-1)^1,\ 2^1(2^n-1)^1,\ 2^2(2^n-1)^1,\ \dots,\ 2^{n-1}(2^n-1)^1. \longleftarrow \text{2nd row}$$

Rewriting these we get these divisors of x:

$$\underbrace{1,\ 2^1,\ 2^2,\ \dots,\ 2^{n-1}}_{\text{1st row}},\ \underbrace{(2^n-1),\ 2^1(2^n-1),\ 2^2(2^n-1),\ \dots,\ \overbrace{2^{n-1}(2^n-1)}^{\boxed{\text{This is } x}}}_{\text{2nd row}}.$$

To see that x is perfect we need to sum the divisors of x (not including x) and show that sum equals x.

Note we excluded $x = 2^{n-1} p$

Let $p = 2^n - 1$. Then, the sum of these divisors is

$$2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} + p + 2^1 p + 2^2 p + \cdots + 2^{n-2} p$$

$$= \frac{2^n - 1}{2-1} + p\left(1 + 2^1 + 2^2 + \cdots + 2^{n-2}\right) = \left(\frac{2^n - 1}{2-1}\right) + p\left(\frac{2^{n-1} - 1}{2-1}\right)$$

$\boxed{\text{If } x \neq 1, \text{ then } 1 + x + x^2 + \cdots + x^k = \frac{x^{k+1} - 1}{x - 1}}$

side example

$x = 6 = 2 \cdot 3$

divisors of x

$2^0, 2^1, 2^0 \cdot 3, \boxed{2^1 \cdot 3}$ x

$2^0 + 2^1 + 2^0 \cdot 3 = 6$

$$= (2^n - 1) + (2^n - 1)(2^{n-1} - 1)$$

$$= (2^n - 1)\left[1 + (2^{n-1} - 1)\right]$$

$$= (2^n - 1) \cdot 2^{n-1}$$

$$= 2^{n-1}(2^n - 1) = x$$

So, x is perfect. ∎

Euclid showed:

If $2^n - 1$ is prime
then $x = 2^{n-1}(2^n - 1)$ is
an even perfect number.

Is the converse true?
Is every even perfect number
of this form? It took
till the time of Euler to solve this. } 1700s

u.kle:.dɛ:s];
m from
e "founder
a during the
uential
teaching
il the late
heorems of
uclid also
ber theory,

Εὐκλείδης,

Euclid

Eukleides of Alexandria

Born	Mid-4th century BC
Died	Mid-3rd century BC
Known for	Euclidean geometry Euclid's *Elements* Euclidean algorithm List of topics named after Euclid

Scientific career

Fields	Mathematics

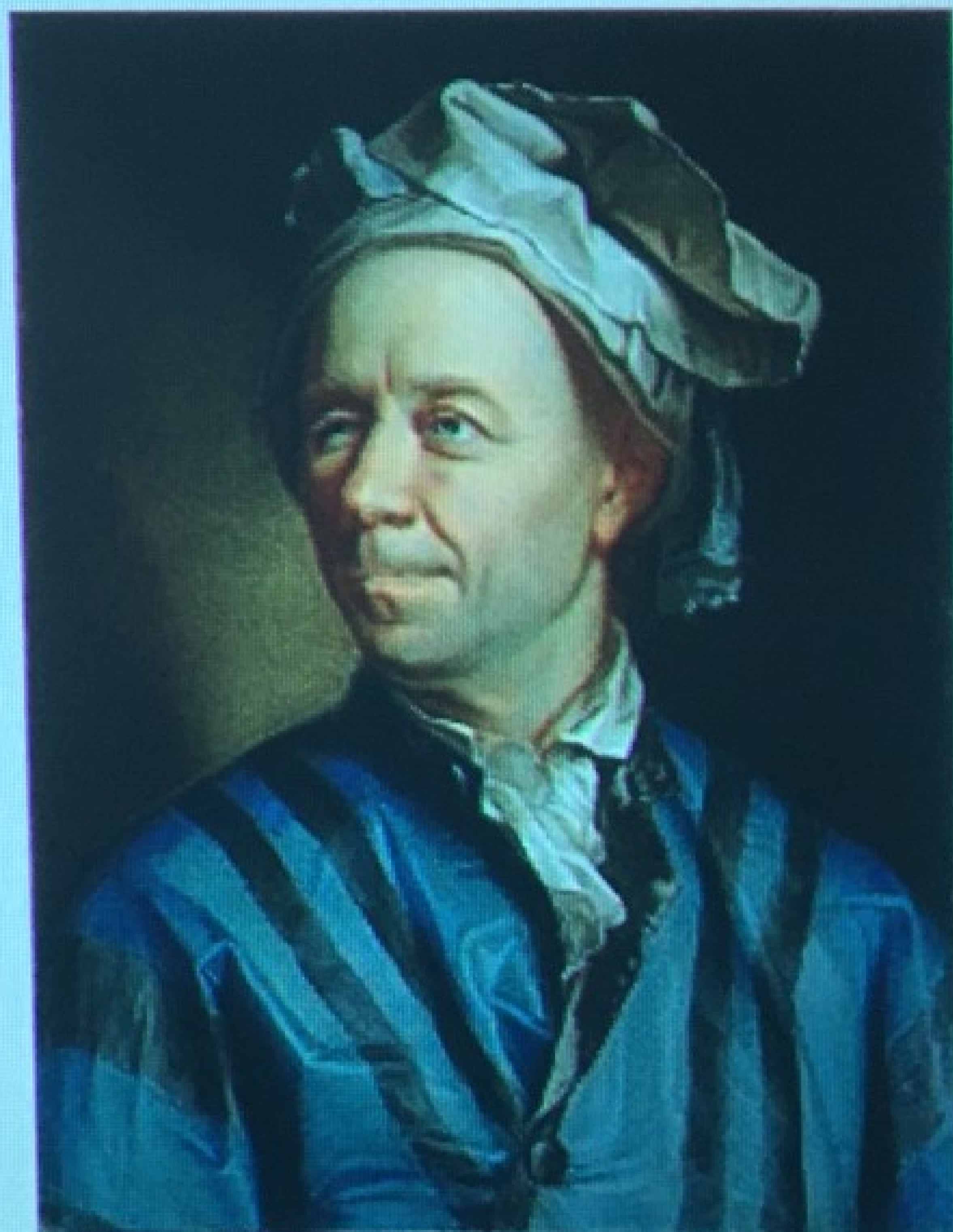
ı).

707 – 18
ɛr,
tial
calculus and
branches
ıuch of the
hematical
o known for
c theory.[4]

tury and is
ed to be the
olumes,[5]
n Saint

influence on
][7]

Leonhard Euler



Portrait by Jakob Emanuel Handmann (1753)

Born	15 April 1707 Basel, Switzerland
Died	18 September 1783 (aged 76) [OS: 7 September 1783] Saint Petersburg, Russian Empire
Alma mater	University of Basel (MPhil)
Known for	See full list
Spouse(s)	Katharina Gsell (1734-1773) Salome Abigail Gsell (1776-1783)

Scientific career

Notation 30

Let $\mathbb{Z}_+ = \{1, 2, 3, 4, \ldots\}$ be the set of positive integers.

Def 31: Define the function $\sigma : \mathbb{Z}_+ \longrightarrow \mathbb{Z}_+$

by $\sigma(n) = \sum_{d \mid n} d$.

Ex 32:

$$\sigma(1) = \sum_{d|1} d = 1$$

$$\sigma(2) = \sum_{d|2} d = 1 + 2 = 3$$

$$\sigma(6) = \sum_{d|6} d = 1 + 2 + 3 + 6 = 12$$

Fact 33:

n is perfect iff

$$\sigma(n) = 2n$$

Ex 34:

$$\sigma(12) = \boxed{\sigma(2^2 \cdot 3)}$$

$$= 2^0 \cdot 3^0 + 2^1 \cdot 3^0 + 2^2 \cdot 3^0$$
$$+ 2^0 \cdot 3^1 + 2^1 \cdot 3^1 + 2^2 \cdot 3^1$$

$$= (2^0 + 2^1 + 2^2) \cdot 3^0 + (2^0 + 2^1 + 2^2) \cdot 3^1$$

$$= \underbrace{(2^0 + 2^1 + 2^2)}_{\sigma(2^2)} \underbrace{(3^0 + 3^1)}_{\sigma(3^1)} = \boxed{\sigma(2^2)\sigma(3)}$$