California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Linear Analysis **Spring 2017**
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Do five of the following seven problems. All problems count equally. If you attempt more than five, the best five will be used.

Please

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

MISCELLANEOUS FACTS

$\sin(a+b) = \sin a \cos b + \cos a \sin b$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$2\sin a \sin b = \cos(a-b) - \cos(a+b)$

$2\cos a \cos b = \cos(a-b) + \cos(a+b)$

$2\sin a \cos b = \sin(a+b) + \sin(a-b)$

$2\cos a \sin b = \sin(a+b) - \sin(a-b)$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax)$$

$$\int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

$$\int x\sin bx\,dx = \frac{1}{b^2}\sin bx - \frac{x}{b}\cos bx$$

$$\int x\cos bx\,dx = \frac{1}{b^2}\cos bx + \frac{x}{b}\sin bx$$

Start of Exam

Spring 2017 # 1. For each of the following sets, determine whether it is a vector space over $\mathbb{R}$. Give reasons for your answers. You may assume the the set of all functions from $\mathbb{R}$ to $\mathbb{R}$ with the operations

$$(f+g)(x) = f(x) + g(x) \quad \text{and} \quad (\lambda f)(x) = \lambda f(x)$$

is a vector space over $\mathbb{R}$.

a. $A = \{f : \mathbb{R} \to \mathbb{R} \mid f(0) = 5\}$ **b.** $B = \{f : \mathbb{R} \to \mathbb{R} \mid f(5) = 0\}$

Spring 2017 # 2. Define $f\colon [-\pi, \pi] \to \mathbb{R}$ by $f(x) = |x|$.

a. Find the Fourier series of f on the interval $[-\pi, \pi]$. (You may use either the exponential or the trigonometric form.)

b. Use your answer from (a) to show that $\displaystyle\sum_{j=1}^{\infty} \frac{1}{(2j-1)^4} = \frac{\pi^4}{96}$.

Spring 2017 # 3. Let $\lambda \in \mathbb{R}$. For each continuous real-valued function f on the interval $[0, 1]$, define Tf on $[0, 1]$ by $(Tf)(x) = x + \lambda \displaystyle\int_0^x f(t)\sin(2\pi t)\,dt$

a. Find a range of values for the parameter λ for which the transformation T is a contraction with respect to the supremum norm $\|\, f\,\|_\infty = \sup\{|\, f(x)\,| : x \in [0, 1]\}$

b. Describe the iterative process for solving the integral equation $f = Tf$ by specifying the transformation to be iterated and explaining how and why this leads to a solution. With $f_0(x) = 0$ for all x as the starting function, compute the first two iterates, $f_1(x)$, and $f_2(x)$.

Spring 2017 # 4. Let $\mathcal{P}_1$ be the space of all polynomials of degree no more than 1 with real coefficients.

a. Find a basis for $\mathcal{P}_1$ which is orthonormal with respect to the inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$.

b. Use the results of part (a) to find constants a and b which make the quantity $\int_0^1 |e^t - at - b|^2\, dt$ as small as possible.

Spring 2017 # 5. Let V be the vector space of all continuous real-valued functions on $[0, 3]$, with the L^∞ norm, $\|\, f\,\|_\infty = \sup\{|\, f(x)\,| : x \in [0, 3]\}$. Let $W = \{f \in V \mid f(1) = f(2)\}$.

a. Find a bounded linear functional $\phi\colon V \to \mathbb{R}$ such that W is the kernel of ϕ, and prove that your answer is correct. (This includes showing that your functional is linear and that it is bounded.)

b. Prove that W is a closed linear subspace of V.

Spring 2017 # 6. Let H be an inner product space. Let A and B be subsets of H. Prove that $(A \cup B)^\perp = A^\perp \cap B^\perp$.

Spring 2017 # 7. Let $e_n(x) = e^{inx}$. Recall that $\{e_n \mid n \in \mathbb{Z}\}$ is an orthonormal basis for the Hilbert space $L^2([0, 1])$, with the appropriate inner product. Let $f \in L^2([0, 1])$. Prove that

$$\sum_{n=0}^{\infty} \frac{n}{n+1} \langle f, e_n\rangle e_n$$

converges in $L^2([0, 1])$ (That is, with respect to the norm of $L^2([0, 1])$).

End of Exam