California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Linear Analysis Spring 2016
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Do five **of the following seven problems. All problems count equally. If you attempt more than five, the best five will be used.**

Please

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

MISCELLANEOUS FACTS

$\sin(a+b) = \sin a \cos b + \cos a \sin b$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$2 \sin a \sin b = \cos(a-b) - \cos(a+b)$

$2 \cos a \cos b = \cos(a-b) + \cos(a+b)$

$2 \sin a \cos b = \sin(a+b) + \sin(a-b)$

$2 \cos a \sin b = \sin(a+b) - \sin(a-b)$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax)$$

$$\int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$$

$$\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

$$\int x\sin bx\,dx = \frac{1}{b^2}\sin bx - \frac{x}{b}\cos bx$$

$$\int x\cos bx\,dx = \frac{1}{b^2}\cos bx + \frac{x}{b}\sin bx$$

Spring 2016 # 1.

a. Let $\mathcal{V}$ be a vector space. Give a definition of what it means for a set L in $\mathcal{V}$ to be *linearly independent.* (Note, the vector space $\mathcal{V}$ may be infinite dimensional and L might be an infinite set). (If your definition is in terms of linear dependence, be sure to define what that means.)

b. Show that the functions $f_0(x) = 1$, $f_1(x) = \cos(x)$ and $f_3(x) = \sin(x)$ are linearly independent (in the vector space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$).

c. Show that the functions $g_0(x) = 1+x$, $g_1(x) = x^2+1$ and $g_3(x) = x^2+2x+3$ are *not* linearly independent (in the vector space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$).

Spring 2016 # 2. Let

$$f(x) = \begin{cases} 0, & \text{for } -\pi \leq x < -\pi/2 \\ 1, & \text{for } -\pi/2 \leq x \leq \pi/2 \\ 0, & \text{for } \pi/2 < x \leq \pi \end{cases}$$

a. Find the Fourier series for f on the interval $[-\pi, \pi]$. (You may use either the exponential or the trigonometric form.)

b. Use the result in part (i) to show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots = \frac{\pi^2}{8}.$$

Spring 2016 # 3. For each of the following sets decide if it is a vector subspace of the vector space of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Justify your answers.

a. The set A of all polynomials p with real coefficients and degree $p = 2$.

b. The set B of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with $f(3) = 0$.

c. The set C of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ with $f(0) = 3$.

Spring 2016 # 4. For each continuous function f on the interval $[0, 1]$, define a continuous function Tf by

$$(Tf)(x) = 1 + \lambda \int_0^x tf(t)dt.$$

a. Find a range of values for the parameter λ for which the transformation T is a contraction with respect to the supremum norm given by $\|g\|_\infty = \sup\{|g(x)| \colon x \in [0, 1]\}$. Justify your answer.

b. Describe the iteration process for solving the integral equation

$$f(x) = 1 + \frac{1}{2} \int_0^x tf(t)dt$$

specifying the transformation to be iterated and explaining how this leads to a solution. State the conclusion offered by the Banach fixed point theorem (the contraction mapping principle) and why that theorem applies.

c. With $f_0(x) = 0$ for all x as the starting function, compute the iterates, $f_1(x)$, $f_2(x)$ and $f_3(x)$.

Spring 2016 # 5. Let $T\colon \mathcal{H} \to \mathcal{H}$ be a bounded linear operator on a Hilbert space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$. Let

$$\ker(T) = \{h \in \mathcal{H}\colon Th = 0\} \text{ and}$$
$$\operatorname{ran}(T) = \{Th\colon h \in \mathcal{H}\}.$$

a. Show that $\ker(T)$ and $\operatorname{ran}(T)$ are vector subspaces of $\mathcal{H}$.
b. Show that $\ker(T)$ is a closed subset of $\mathcal{H}$.
c. Show that if $\langle Th, k \rangle = \langle h, Tk \rangle$ for all h and k in $\mathcal{H}$, then $\ker(T) = (\operatorname{ran}(T))^{\perp}$

Spring 2016 # 6. Let $C([-1,1])$ be the vector space of all continuous functions from $[-1,1]$ to $\mathbb{C}$. We consider $C([-1,1])$ as a normed vector space, with the supremum norm

$$\| f \|_{\infty} = \sup\{| f(x) | : \ x \in [-1,1]\}.$$

Define a function $\phi\colon C([-1,1]) \to \mathbb{C}$ by $\phi(f) = f(1/2)$.

a. Show that ϕ is a linear map.
b. Show that ϕ is continuous.
c. Find $\| \phi \|$, the operator norm of ϕ.

Note: parts (ii) and (iii) can be done in either order.

Spring 2016 # 7. Let $\mathcal{P}^1$ be the space of all polynomials with real coefficients and degree no more than 1 with the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$.

a. Find an orthonormal basis for $\mathcal{P}^1$ with respect to the inner product $\langle \cdot, \cdot \rangle$.
b. Use part (i) to find constants a and b such that

$$\int_0^1 (ax + b - x^2)^2 dx$$

is as small as possible.

End of Exam