California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Linear Analysis **Spring 2014**
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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

MISCELLANEOUS FACTS

$\sin(a+b) = \sin a \cos b + \cos a \sin b$ $\qquad$ $\cos(a+b) = \cos a \cos b - \sin a \sin b$

$2\sin a \sin b = \cos(a-b) - \cos(a+b)$ $\qquad$ $2\cos a \cos b = \cos(a-b) + \cos(a+b)$

$2\sin a \cos b = \sin(a+b) + \sin(a-b)$ $\qquad$ $2\cos a \sin b = \sin(a+b) - \sin(a-b)$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} \qquad \int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

$$\int x\sin bx\,dx = \frac{1}{b^2}\sin bx - \frac{x}{b}\cos bx \qquad \int x\cos bx\,dx = \frac{1}{b^2}\cos bx + \frac{x}{b}\sin bx$$

Spring 2014 # 1. Let $f : \mathbb{R} \to \mathbb{R}$ be the 2π-periodic function given for $-\pi \le x < \pi$ by $f(x) = |\, x \,|$.

a. Compute the Fourier series for f. (Either exponential or trigonometric form, your choice).

b. Give a statement of the Parseval identity and compute what it tells you for the example in part **a**.

Spring 2014 # 2. Let $\mathcal{X}$ be the space $C([0, 2]; \mathbb{R})$ of all continuous real valued functions on the interval $[a, b]$ equipped with the uniform norm, $\|\, f \,\|_\infty = \sup\{|\, f(t) \,| : t \in [0, 2]\}$.

a. Find a range of values for the numerical parameter λ for which the transformation $T : \mathcal{X} \to \mathcal{X}$ given for f in $\mathcal{X}$ and x in $[0, 2]$ by

$$(Tf)(x) = x + \lambda \int_0^x t^2 f(t)\, dt$$

is a contraction on $\mathcal{X}$.

b. Describe the iterative process for solving the integral equation

$$\text{(VIE)} \qquad f(x) = x + \frac{1}{4} \int_0^x t^2 f(t)\, dt$$

specifying the transformation to be iterated and explaining how it leads to a solution. With $f_0(x) = 0$ for all x, compute the first two iterates $f_1(x)$ and $f_2(x)$.

c. Show that a function f is a solution to (VIE) if and only if it is also a solution to an initial value problem of the form

$$\text{(IVP)} \qquad f'(x) + p(x) f(x) = \phi(x) \quad \text{with } f(0) = A.$$

stating explicitly what $p(x)$, $\phi(x)$, and A are in this case.

Spring 2014 # 3. Let $\mathcal{X}$ be the space $C([0, 2\pi]; \mathbb{R})$ of all continuous real valued functions on the interval $[0, 2\pi]$ equipped with the uniform norm, $\|\, f \,\|_\infty = \sup\{|\, f(t) \,| : t \in [0, 2\pi]\}$.

Let $\mathcal{Y}$ be the subspace $C^1([0, 2\pi]; \mathbb{R})$ consisting of those f in $\mathcal{X}$ which are differentiable with f' also continuous.

Let $D : \mathcal{Y} \to \mathcal{X}$ be the linear transformation given by $Df = f'$.

a. Show that D is not continuous if the norm $\|\cdot\|_\infty$ is used on both domain and range. That is, as a transformation $D : (\mathcal{Y}, \|\cdot\|_\infty) \to (\mathcal{X}, \|\cdot\|_\infty)$

b. Show that the formula $[[f]] = \|\, f \,\|_\infty + \|\, f' \,\|_\infty$ gives a norm on $\mathcal{Y}$. (You may assume that $\|\cdot\|_\infty$ is a norm.)

c. Show that D is continuous as a transformation $D : (\mathcal{Y}, [[\cdot]]) \to (\mathcal{X}, \|\cdot\|_\infty)$

Suggestion: For part **a** the functions $f_k(x) = \sin kx$, $k = 1, 2, 3, \ldots$ might be useful.

Spring 2014 # 4. Let $\mathcal{P}_1$ be the space of all polynomials of degree no more than 1.

a. Find a basis for $\mathcal{P}_1$ which is orthonormal with respect to the inner product $\langle f, g \rangle = \int_0^1 f(t)\overline{g(t)}\,dt$.

b. Use the results of part **a** to find constants a and b which make the quantity $J = \int_0^1 \left| a + bt - \sqrt{t} \right|^2 \, dt$ as small as possible.

Spring 2014 # 5. Each of the following is a vector space over $\mathbb{R}$. For each, determine the dimension over $\mathbb{R}$ and justify your answer. (if possible, find and display a basis.)

a. $A =$ The set of all vectors (x, y, z) in $\mathbb{R}^3$ such that

$$x - 2z = 0 \quad \text{and} \quad y - 3z = 0$$

b. $B =$ The space of all 2×2 matrices with real entries and trace 0.

(The trace of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the sum of the main diagonal entries, $a + d$.)

Spring 2014 # 6. Let $\phi : C([-1, 1]) \to \mathbb{C}$ be defined by $\phi(f) = f(0) + f(1/2)$ for each continuous function f.

a. Show that ϕ is linear.

b Show that ϕ is continuous if the supremum norm, $\| f \|_\infty = \sup\{| f(t) | \mid t \in [-1, 1]\}$ is used on $C([-1, 1])$. (The "norm" on $\mathbb{C}$ is absolute value.)

c. Find a constant B such that $\| \phi \|_{\text{op}} \le B$. ($\| \phi \|_{\text{op}}$ is the "operator" or "functional" norm of ϕ.)

(You may do parts **b** and **c** in either order.)

Spring 2014 # 7. For each of the following decide if it is a vector space over $\mathbb{R}$. Give reasons for your answers. (You may assume that the set of all real valued functions on the interval $[0, 1]$ is a vector space with the operations $(f + g)(x) = f(x) + g(x)$ and $(\lambda f)(x) = \lambda f(x)$.)

a. $A =$ The set of all polynomials in one variable with real coefficients having degree exactly equal to 3 .

b. $B =$ The set of continuous real valued functions on the interval $[0, 1]$ such that $f(0) = 0$ and $f(1) = 0$

c. $C =$ The set of continuous real valued functions on the interval $[0, 1]$ such that $f(0) = 0$ and $f(1) = 1$

End of Exam