California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Linear Analysis   Spring 2008

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Do five of the following eight problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\bar{z}$ denotes the complex conjugate of the complex number z.
$|\, z\,|$ denotes the absolute value of the complex number z

$\mathcal{C}([a,b])$ denotes the space of all continuous functions on the interval $[a,b]$. If there is need to specify the possible values, $\mathcal{C}([a,b],\mathbb{R})$ will denote the space of all continuous real valued functions on $[a,b]$ and $\mathcal{C}([a,b],\mathbb{C})$ the space of all continuous complex valued functions.

$L^2([a,b])$ denotes the space of all functions on the inteval $[a,b]$ such that $\int_a^b |\, f(x)\,|^2 \; dx < \infty$

MISCELLANEOUS FACTS

$\sin(a+b) = \sin a\cos b + \cos a \sin b$ $\cos(a+b) = \cos a\cos b - \sin a\sin b$

$2\sin a\sin b = \cos(a-b) - \cos(a+b)$ $2\cos a\cos b = \cos(a-b) + \cos(a+b)$

$2\sin a\cos b = \sin(a+b) + \sin(a-b)$ $2\cos a\sin b = \sin(a+b) - \sin(a-b)$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} \qquad \int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

Spring 2008 # 1. For each of the following decide if it is a vector space over $\mathbb{R}$. Give reasons for your answers. (You may assume that the set of all real valued functions on the interval $[0,1]$ is a vector space with the operations $(f+g)(x) = f(x) + g(x)$ and $(\lambda f)(x) = \lambda f(x)$.)

a. $A = \{f : [0,1] \to \mathbb{R} : \int_0^1 |\, f(x)\,|\, dx = 0\}$
b. $B = \{f : [0,1] \to \mathbb{R} : f'(x) + 4f(x) = 0 \text{ and } f(0) = 1\}$
c. $C = \{f : [0,1] \to \mathbb{R} : \int_0^1 f(x)\, dx = 1\}$
d. $D = \{f : [0,1] \to \mathbb{R} : f'(x) + 4f(x) = 0\}$

Spring 2008 # 2. Let $\mathcal{P}^1$ be the space of all polynomials with real coefficients and degree no more than 1 with the inner product $\langle\, f, g\,\rangle = \int_0^1 f(t)g(t)\, dt$.

For each p in $\mathcal{P}^1$, let $\phi(p) = p(0)$

a. Find a polynomial q in $\mathcal{P}^1$ such that $\phi(p) = \langle\, p, q\,\rangle$ for all p in $\mathcal{P}^1$.
b. Find the norm, $\|\,\phi\,\|$, of ϕ as a bounded linear functional on $\mathcal{P}^1$. (Give reasons for your answer, explain why your method works.)

Spring 2008 # 3. For each continuous function f on $[0,1]$, define Kf by

$$(Kf)(x) = \int_0^1 (1 + xt) f(t)\, dt.$$

a. Find all nonzero eigenvalues of the operator K.
b. Find a function f such that $f(x) = -3 + (Kf)(x)$.

Spring 2008 # 4. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator whose matrix with respect to the standard basis is $\dfrac{1}{4}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$.

a. Find all eigenvalues of T
b. For each value found in part (a), find the corresponding set of eignevectors.
c. Find an orthonormal basis for $\mathbb{R}^2$ which consists of eigenvectors for T.
d. Show that $\lim_{n\to\infty} \|\, T^n v\,\| = 0$ for each vector v in $\mathbb{R}^2$.
e. Find the operator norm of T (Give reasons for your answer.)

Spring 2008 # 5. Let $\mathcal{H}$ be the space of (piecewise) continuous 2π-periodic functions on the real line. For f and g in $\mathcal{H}$, consider the inner product $\langle\, f, g\,\rangle = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} f(x)\overline{g(x)}\, dx$ and the functions $e_n(x) = e^{inx}$ for integer n with $-\infty < n < \infty$.

a) (3 pts.) Show that the family of functions $\{e_n \mid -\infty < n < \infty\}$ is an orthonormal family in $\mathcal{H}$.

b) (3 pts.) Discuss briefly what it means for the family $\{e_n \mid -\infty < n < \infty\}$ to be a complete orthonormal family (more or less equivalently, an orthonormal basis for $L^2([-\pi,\pi])$). Say something if you can about why you know it is one.

c) (7 pts.) Let $f(x) = x$ for $-\pi \le x < \pi$, and assume f is extended to be a 2π-periodic function. Compute the Fourier series of f with respect to the orthonormal sequence $\{e_n \mid -\infty < n < \infty\}$.

d) (7 pts.) Use parts **(a)**, **(b)**, and **(c)** to show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Spring 2008 # 6. Let ℓ^2 be the space of square-summable sequences

$$x = (x_1, x_2, x_3, \dots) \qquad \text{with } \sum_{k=1}^{\infty} |\, x_k \,|^2 < \infty.$$

Let $\mathcal{B} = \{e_k\}_{k=1}^{\infty}$ be the standard basis for ℓ^2 given by

$$\begin{aligned} e_1 &= (1, 0, 0, 0, 0, \dots) \\ e_2 &= (0, 1, 0, 0, 0, 0, \dots) \\ e_3 &= (0, 0, 1, 0, 0, 0, \dots) \\ &\vdots \end{aligned}$$

For $k = 1, 2, 3, \dots$, let $Te_k = \frac{1}{k} e_{k+1}$

a. Show that T defines a bounded linear operator on ℓ^2, and find the operator norm, $\|\, T \,\|$, of T.

b. Find all the eigenvalues of T or show that there are none.

Spring 2008 # 7. For each f in the space $C([0, 2])$ of all continuous functions on the ijnterval $[0, 2]$ and each number λ, let Tf be the function on $[0, 2]$ defined by $(Tf)(x) = x + \lambda \int_0^x (x - t) f(t)\, dt$.

a. Show that f is a solution to the integral equation

$$(VIE) \qquad f(x) = x + \lambda \int_0^x (x - t) f(t)\, dt$$

if and only if it is a solution to the initial value problem

$$(IVP) \qquad f''(x) = \lambda f(x) \quad \text{with } f(0) = 0 \text{ and } f'(0) = 1.$$

b. Find a range of values for the parameter λ for which the transformation T is a contraction on $C([0, 2])$ with respect to the supremum norm $\|\, f \,\|_\infty = \sup_{x \in [0,2]} |\, f(x) \,|$. Justify your answer.

b. Describe the iterative process for solving the integral equation (VIE) specifying the transformation to be iterated and explaining why this leads to a solution. With $f_0(x) = 0$ for all x as the starting function, compute the iterates, $f_1(x)$and $f_2(x)$.

Spring 2008 # 8. Suppose $k(x, y)$ is a continuous real valued function on the square $[0, 1] \times [0, 1]$. For f in the space $L^2([0, 1])$ of square integrable functions on $[0, 1]$, let Tf be defined by $(Tf)(x) = \int_0^1 k(x, t) f(t)\, dt$. With the inner product $\langle\, f, g \,\rangle = \int_0^1 f(t) \overline{g(t)}\, dt$,

a. Show that T is a bounded linear operator on $L^2([0, 1])$.

b. Show that the adjoint is given by $(T^* f)(x) = \int_0^1 \overline{k(t, x)} f(t)\, dt$.

End of Exam