California State University – Los Angeles
Department of Mathematics
Master's Degree Comprehensive Examination

Linear Analysis   Spring 2003

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Do **five** of the following eight problems. Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.
$\mathbb{R}$ denotes the set of real numbers.
$\mathrm{Re}(z)$ denotes the real part of the complex number z.
$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.
$\bar{z}$ denotes the complex conjugate of the complex number z.
$|z|$ denotes the absolute value of the complex number z
$\mathcal{C}([a,b])$ denotes the space of all continuous functions on the interval $[a,b]$. If there is need to specify the possible values, $\mathcal{C}([a,b],\mathbb{R})$ will denote the space of all continuous real valued functions on $[a,b]$ and $\mathcal{C}([a,b],\mathbb{C})$ the space of all continuous complex valued functions.
$L^2([a,b])$ denotes the space of all functions on the inteval $[a,b]$ such that $\int_a^b |f(x)|^2\ dx < \infty$

MISCELLANEOUS FACTS

$$\sin(a+b) = \sin a \cos b + \cos a \sin b \qquad \cos(a+b) = \cos a \cos b - \sin a \sin b$$

$$2\sin a \sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a \cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a \cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a \sin b = \sin(a+b) - \sin(a-b)$$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int x\sin(ax)\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax) \qquad \int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$$

Spring 2003 # 1. Which of the following are vector spaces over $\mathbb{R}$? For each explain why or why not. (You may assume that the spaces $C([0,1],\mathbb{R})$ of all continuous real valued functions on $[0,1]$ and the space $\mathcal{P}_{\mathbb{R}}$ of all polynomials with real coefficients are vector spaces over $\mathbb{R}$.)

a. $A = \{f \in C([0,1],\mathbb{R}) : \int_0^1 f(t)\,dt = 0\}$
b. $B = \{f \in C([0,1],\mathbb{R}) : f''$ exists , $f''(x) + f(x) = 0$ for all x in $\mathbb{R}$, and $f(0) = 1\}$
c. $C = \{p \in \mathcal{P}_{\mathbb{R}} : \text{degree}(p) \le 3,$ including the zero polynomial $\}$
d. $D = \{p \in \mathcal{P}_{\mathbb{R}} : \text{degree}(p) \le 3,$ or $p = 0\}$

Spring 2003 # 2. **a.** Define what it means for a set $\mathcal{L}$ of vectors in a vector space $\mathcal{V}$ to be linearly independent. (Be careful, your definition must allow for the possibility that $\mathcal{L}$ is an infinite set.)

b. Show that the functions $f_0(x) = 1$, $f_1(x) = e^x$, and $f_2(x) = e^{2x}$ are linearly independent as functions on $\mathbb{R}$. (Hint: $e^{2x} = (e^x)^2$.)

c. Show that the functions $g_0(x) = 1$, $g_1(x) = 2\cos^2 x$, and $g_2(x) = 3\sin^2 x$ are not linearly independent as functions on $\mathbb{R}$.

Spring 2003 # 3. Let $C([-1,1])$ be the space of continuous functions on $[-1,1]$ with the inner product $\langle F, G \rangle = \int_{-1}^1 F(t)\overline{G(t)}\,dt$. (You may assume this is an inner product.) Let $\|\cdot\|_2$ be the associated norm on $C([-1,1])$.

Let $\mathcal{H}$ be the space of all functions f on $[-1,1]$ with f' continuous on $[-1,1]$. For f and g in $\mathcal{H}$, put

$$[f,g] = f(0)\overline{g(0)} + \int_{-1}^1 f'(t)\overline{g'(t)}\,dt$$

a. Show that $[f,g]$ gives an inner product on $\mathcal{H}$.
(Suggestion: For what f is $\langle f', f' \rangle = 0$?)

b. Let $\|\cdot\|_1$ be the norm on $\mathcal{H}$ associated with the inner product $[\cdot,\cdot]$. For f in $\mathcal{H}$, let $Df = f'$.

(i) Show that D is a linear operator from $\mathcal{H}$ into $C([-1,1])$.
(ii) Show that D is bounded as a linear operator with the norm $\|\cdot\|_1$ on $\mathcal{H}$ and $\|\cdot\|_2$ on $C([-1,1])$.

Spring 2003 # 4. Let L be the differential operator $L = -d^2/dx^2$.

a. Find a function $G(x,t)$ such that for given continuous functions f on $[0,1]$ solutions to the boundary value problem

$$Ly = f \qquad \text{with } y(0) = y'(0) \text{ and } y(1) = y'(1)$$

are given by

$$y(x) = \int_0^1 G(x,t)f(t)\,dt.$$

b. Use the function G found in part **a** to solve the problem when $f(x) = x$.

Spring 2003 # 5. Consider the integral equation

$$f(x) = g(x) + \lambda \int_0^1 x^3 t^3 f(t)\, dt \tag{IE}$$

where λ is a numerical parameter.

a. With $g(x) = x$, solve for $f(x)$ by using the Neumann series. That is, by computing $g + \sum_{n=1}^{\infty} \lambda^n K^n g$ for an appropriate operator K.

b. Find a function $R(x, t, \lambda)$ such that solutions to the equation (IE) are given for each g by

$$f(x) = g(x) + \lambda \int_0^1 R(x, t, \lambda) g(t)\, dt$$

c. Take $g(x) = x$ in part **b** and show that one gets the same solution as in part **a**.

Spring 2003 # 6. Let $f_1,\ f_2,\ f_3,\ \ldots$ be a sequence of continuous functions on $[0, 1]$, and f be a continuous function on $[0, 1]$.

a. Define each of the following.

(i) The sequence f_n converges pointwise to f on $[0, 1]$.
(ii) The sequence f_n converges uniformly to f on $[0, 1]$.
(iii) The sequence f_n converges in L^2-norm to f on $[0, 1]$.

b. Show that if the sequence f_n converges in L^2-norm to f on $[0, 1]$, then

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx = \int_0^1 f(x)\, dx.$$

c. Give an example of a sequence $\{f_n\}_{n=1}^{\infty}$ of continuous functions on $[0, 1]$ and a limit function f such that the sequence f_n converges pointwise to f on $[0, 1]$ but

$$\lim_{n\to\infty} \int_0^1 f_n(x)\, dx \neq \int_0^1 f(x)\, dx.$$

Spring 2003 # 7. Let $L^2([-\pi,\pi])$ have the inner product
$\langle f, g \rangle = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} f(x)\overline{g(x)}\,dx.$

a. Find the Fourier series for the function

$$f(x) = \begin{cases} 0, & \text{for } -\pi \le x < 0 \\ 1, & \text{for } 0 \le x \le \pi \end{cases}$$

with respect to the orthonormal basis $\{e_n(x) = e^{inx} : n = 0, \pm 1, \pm 2, \pm 3\}$

b. Use the result of part **a** to show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}.$$

Spring 2003 # 8. Supppose g is a continuous function on $[0,1]$. For f in $C([0,1])$ define Tf by

$$(Tf)(x) = g(x) + \lambda \int_0^1 e^{x-t} f(t)\,dt.$$

a. Find a range of values of λ for which T is a contraction with respect to the supremum norm on $C([0,1])$.

b. Find a range of values of λ for which T is a contraction with respect to the L^2-norm on $C([0,1])$.

c. Describe the iterative process for finding a solution f to the equation

$$f(x) = g(x) + \lambda \int_0^1 e^{x-t} f(t)\,dt$$

explaining how the procedure works and how one knows that it leads to a solution.

d. With $g(x) = e^x$ and $f_0(x) = 0$ for all x, compute the first three iterates, f_1, f_2, and f_3.

End of Exam