CSULA Mathematics
Masters Degree Comprehensive Examination

Linear Analysis Fall 2018 Gutarts, Hajaiej , Hoffman*

Do at least five of the following eight problems. All problems count equally. If you attempt more than five, the best five will be used.

Please

(1) Write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

MISCELLANEOUS FACTS

$\sin(a+b) = \sin a \cos b + \cos a \sin b$ $\cos(a+b) = \cos a \cos b - \sin a \sin b$

$2\sin a \sin b = \cos(a-b) - \cos(a+b)$ $2\cos a \cos b = \cos(a-b) + \cos(a+b)$

$2\sin a \cos b = \sin(a+b) + \sin(a-b)$ $2\cos a \sin b = \sin(a+b) - \sin(a-b)$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} \qquad \int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

Fall 2018 # 1. **a.** State and prove the parallelogram law for inner product spaces.

b. Show that there is no inner product on the space $C([0,1])$ of all continuous real valued function on the interval $[0,1]$ for which the supremum norm, $\| f \|_\infty = \sup\{| f(t) | : t \in [0,1]\}$, is the associated norm.

Fall 2018 # 2. For each of the following decide if it is a vector subspace of the vector space $\mathcal{V}$ of all seqences $\vec{x} = (x_1, x_2, x_3, \dots)$ of real numbers. (Vector addition and scalar multiplication are defined as usual for sequences.

$$(x_1, x_2, \dots) + (y_1, y_2, \dots) = (x_1+y_1, x_2+y_2, \dots) \qquad \lambda(x_1, x_2, \dots) = (\lambda x_1, \lambda x_2, \dots).$$

a. $A = \{\vec{x} \in \mathcal{V} : x_1 x_2 = 0\}$
b. $B = \{\vec{x} \in \mathcal{V} : x_1 + x_2 = 0\}$
c. $C = \{\vec{x} \in \mathcal{V} : x_1 + x_2^2 = 0\}$

Fall 2018 # 3. Suppose A is a bounded linear operator from a Hilbert space $\mathcal{H}$ into itself. with $\| A \| < 1$. (This is the operator norm.) Let g be in $\mathcal{H}$. Show that the equation $f = g + Af$ has a unique solution f in $\mathcal{H}$.

Fall 2018 # 4. Let $\mathcal{M}$ be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$v_1 = (3,0,0,0) \qquad v_2 = (1,0,0,1) \quad \text{and } v_3 = (1,1,1,0).$$

a. Find a basis for $\mathcal{M}$ which is orthonormal with respect to the usual inner product (dot product) on $\mathbb{R}^4$.

b. Find the vector w in $\mathcal{M}$ at minimum distance from $w_o = (4,3,2,1)$.

Fall 2018 # 5. Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation given with respect to the standard basis by the matrix $\begin{pmatrix} 0 & 1/2 & 0 \\ 0 & 0 & 1/2 \\ 0 & 0 & 0 \end{pmatrix}$. Let I be the identity operator on $\mathbb{R}^3$.

a. Find the operator norm of S (with the usual Euclidean norm on $\mathbb{R}^3$). Justify your answer.

b. Use the Neumann series to find the matrix for $(I-S)^{-1}$ with respect to the standard basis for $\mathbb{R}^3$.

Fall 2018 # 6. Let ℓ^2 be the space of square-summable sequences

$$x = (x_1, x_2, x_3, \dots) \qquad \text{with } \sum_{k=1}^{\infty} |\, x_k \,|^2 < \infty.$$

Let $\mathcal{B} = \{e_k\}_{k=1}^{\infty}$ be the standard basis for ℓ^2 given by

$$e_1 = (1,0,0,0,0,\dots) \quad e_2 = (0,1,0,0,0,0,\dots) \quad e_3 = (0,0,1,0,0,0,\dots) \quad \dots$$

For $k = 1,2,3,\dots$, let $Te_k = \frac{1}{k} e_{k+1}$

a. Show that T defines a bounded linear operator on ℓ^2, and find the operator norm, $\|\, T \,\|$, of T.

b. Find all the eigenvalues of T or show that there are none.

Fall 2018 # 7. Each of the following is a vector space over $\mathbb{R}$. (You may assume that and do not need to prove it.) For each, determine the dimension over $\mathbb{R}$ and justify your answer. (For full credit, by displaying a basis.)

a. $A =$ the set of all vectors (x,y,z,t) in $\mathbb{R}^4$ such that $x+y=z+t$.

b. $B =$ the space of all polynomials p with real coefficients and degree no more than 2 satisfying $p(1) = 0$.

Fall 2018 # 8. Let a be a real constant with $0 < a < \pi$.
Put $f(x) = 1$ for $|\, x \,| \le a$ and $f(x) = 0$ for $a < |\, x \,| \le \pi$.

a. Compute the Fourier series for f on $[-\pi, \pi]$. (Trigonometric or exponential, your choice)

b. Show that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} \sin^2 ka = \frac{a(\pi - a)}{2}$.

End of Exam