# California State University – Los Angeles
# Mathematics
# Masters Degree Comprehensive Examination

## Linear Analysis    Fall 2014

**Gutarts*, Hoffman, Krebs**

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**Do five of the following seven problems.**
**If you attempt more than 5, the best 5 will be used.**
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

**Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.**

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**MISCELLANEOUS FACTS**

$\sin(a+b) = \sin a \cos b + \cos a \sin b$    $\cos(a+b) = \cos a \cos b - \sin a \sin b$

$2\sin a \sin b = \cos(a-b) - \cos(a+b)$    $2\cos a \cos b = \cos(a-b) + \cos(a+b)$

$2\sin a \cos b = \sin(a+b) + \sin(a-b)$    $2\cos a \sin b = \sin(a+b) - \sin(a-b)$

$\displaystyle\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax)$    $\displaystyle\int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$

$\displaystyle\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$    $\displaystyle\int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$

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**Fall 2014 # 1.** Let $f : \mathbb{R} \to \mathbb{R}$ be the $2\pi$ periodic funcion given on $0 \le x \le 2\pi$ by $f(x) = 1$ for $0 \le x < \pi$ and $f(x) = 0$ for $\pi \le x < 2\pi$.

**a.** (8pts.) Find the Fourier series for $f$. (Either exponential form or trigonometric for, your choice.)

**b.** (4pts.) Give a statement of any form of the Parseval Identity theorem.

**c.** (8pts.) show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots = \frac{\pi^2}{8}.$$

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**Fall 2014 # 2.** Suppose $\mathcal{E} = \{e_1, e_2, e_3, e_4\}$ is an orthonormal basis for a Hilbert space $\mathcal{H}$ and that $T : \mathcal{H} \to \mathcal{H}$ is a linear operator with

$$Te_1 = e_2; \quad Te_2 = 2e_3; \quad Te_3 = 3e_4; \quad \text{and } Te_4 = 0.$$

**a.** Find the matrix for $T$ with respect to the orthonormal basis $\mathcal{E}$.

**b.** Show that $T$ is bounded as a linear operator on $\mathcal{H}$. (Do not just quote the theorem that every linear operator on a finite dimensional normed space is bounded.)

**c.** Find the operator norm of $T$ giving reasons to justify your answer.

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**Fall 2014 # 3.** Let $P_1$ be the space of all polynomials $ax + b$ of degree no more than 1 with the inner product $\langle f, g \rangle = \int_0^2 f(t)\overline{g(t)}\, dt$. (You may assume this is an inner product.)

**a.** Find a basis for the vector space $P_1$ which is orthonormal with respect to that inner product.

**b.** Find constants $a$ and $b$ which make the quantity $\int_0^2 \left| x^3 - ax - b \right|^2 dx$ as small as possible.

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**Fall 2014 # 4.** For each of the following, determine if it is a vector space over $\mathbb{R}$. Give reasons for your answers. ($C(\mathbb{R}, \mathbb{R})$ is the space of all continuous real valued functions on $\mathbb{R}$ which you may assume is a vector space. You may also assume that $\mathbb{R}^4$ with the usual operations is a vector space.)

**a.** $\mathcal{A} = \{f \in C(\mathbb{R}, \mathbb{R}) : f''$ exists and $f''(x) + 3f'(x) + 5f(x) = 0, \forall x \in \mathbb{R}\}$

**b.** $\mathcal{B} = \{f \in C(\mathbb{R}, \mathbb{R}) : f''$ exists and $f''(x) + 3f'(x) + 5f(x) = \cos x, \forall x \in \mathbb{R}\}$

**c.** $\mathcal{C}$ = the set of all $2 \times 2$ matrices with entries in $\mathbb{R}$ with the operations of matrix addition and scalar multiplication

**d.** $\mathcal{D}$ = the set of all $2 \times 2$ matrices with entries in $\mathbb{R}$ and determinant 0 with the operations of matrix addition and scalar multiplication

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**Fall 2014 # 5.** Let $H$ be a Hilbert space over $\mathbb{C}$ and $S$ and $T$ be bounded linear operators on $H$. Are the following true or false. If true, prove. If false, explain what's missing or give a counterexample, where appropriate.

**a.** (3pts.) $T$ is self-adjoint $\Rightarrow$ $T$ is normal.
**b.** (3pts.) $T$ is unitary $\Rightarrow$ $T$ is normal.
**c.** (3pts.) $T$ is normal $\Rightarrow$ $T$ is self-adjoint.
**d.** (3pts.) $T$ is normal $\Rightarrow$ $T$ is unitary.
**e.** (4pts.) $S$ and $T$ are self-adjoint $\Rightarrow$ the product $ST$ is self-adjoint.
**f.** (4pts.) $S$ and $T$ are unitary $\Rightarrow$ the product $ST$ is unitary.

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**Fall 2014 # 6.** Let $H$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. Suppose $T : H \to H$ is a bounded linear operator such that $\langle Tf, g \rangle = \langle f, Tg \rangle$ for all $f$ and $g$ in $H$.

**a.** Show that all eigenvalues of $T$ are real.
**b.** Show that eigenvectors of $T$ corresponding to different eigenvalues are orthogonal with respect to the inner product $\langle \cdot, \cdot \rangle$.

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**Fall 2014 # 7.** For each continuous function $f$ on the interval $[0, 1]$, define $Tf$ on $[0, 1]$ by $(Tf)(x) = e^x + \lambda \int_0^x e^{x-t} f(t)dt$

**a.** Find a range of values for the parameter $\lambda$ for which the transformation $T$ is a contraction with respect to the supremum norm $\| f \|_\infty = \sup\{| f(t) | : t \in [0, 1]\}$.
**b.** Find a range of values for the parameter $\lambda$ for which the transformation $T$ is a contraction with respect to the $L^2$ norm $\| f \|_2 = \left( \int_0^1 | f(t) |^2 \, dt \right)^{1/2}$.
**c.** Describe the iterative process for solving the integral equation $f(x) = e^x + \lambda \int_0^x e^{x-t} f(t)dt$ specifying the transformation to be iterated and explaining how this leads to a solution. With $f_0(x) = 0$ for all $x$ as the starting function, compute the first three iterates, $f_1(x)$, $f_2(x)$, and $f_3(x)$.

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# End of Exam