California State University – Los Angeles
Mathematics
Masters Degree Comprehensive Examination

Linear Analysis **Fall 2013**

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Do five of the following seven problems.
If you attempt more than 5, the best 5 will be used.
Please

(1) Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.
(2) Write on one side of the paper only
(3) Begin each problem on a new page
(4) Assemble the problems you hand in in numerical order

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

MISCELLANEOUS FACTS

$\sin(a+b) = \sin a \cos b + \cos a \sin b$

$\cos(a+b) = \cos a \cos b - \sin a \sin b$

$2\sin a \sin b = \cos(a-b) - \cos(a+b)$

$2\cos a \cos b = \cos(a-b) + \cos(a+b)$

$2\sin a \cos b = \sin(a+b) + \sin(a-b)$

$2\cos a \sin b = \sin(a+b) - \sin(a-b)$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} \qquad \int e^{ax}\cos bx\,dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}$$

Fall 2013 # 1. Let $\mathcal{M}$ be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$v_1 = (2,0,0,0) \qquad v_2 = (1,0,1,0) \quad \text{and } v_3 = (1,1,0,1).$$

a. Find a basis for $\mathcal{M}$ which is orthonormal with respect to the usual inner product (dot product) on $\mathbb{R}^4$.

b. Find the vector w in $\mathcal{M}$ at minimum distance from $w_o = (1,2,3,4)$.

Fall 2013 # 2. Let a be a real constant with $0 < a < \pi$. Define f on $[-\pi, \pi]$ by putting

$$f(x) = \begin{cases} 0, \text{ for } -\pi < x < -a \\ 1, \text{ for } -a \le x \le a \\ 0, \text{ for } a < x \le \pi \end{cases} .$$

and then extending it to be 2π-periodic on $\mathbb{R}$

a. Compute either the exponential or the trigonometric form of the Fourier series for f. (Your choice which)

b. Use the result of part **a** to show that $\displaystyle\sum_{k=1}^{\infty} \frac{\sin^2(ka)}{k^2} = \frac{a(\pi - a)}{2}$

Fall 2013 # 3. Suppose $\mathcal{H}$ is an inner product space with inner product $\langle \cdot, \cdot \rangle$ and associated norm $\|\cdot\|$. A sequence $\{f_n\}_{n=1}^{\infty}$ is $\mathcal{H}$ is said to converge weakly to a weak limit g in $\mathcal{H}$ (written $f_n \xrightarrow{w} g$) if $\langle f_n, h \rangle \to \langle g, h \rangle$ as numbers for every h in $\mathcal{H}$.

a. Show that if $\| f_n - g \| \to 0$ as $n \to \infty$, then $f_n \xrightarrow{w} g$. (Norm convergence implies weak convergence.)

b. Suppose $e_1, e_2, e_3, \ldots$ is an infinite orthonormal sequence in $\mathcal{H}$. Show that $e_n \xrightarrow{w} 0$. (Suggestion: Consider the Bessel inequality.)

c. Use part **b** to show that in an infinite dimensional inner product space $\mathcal{H}$ weak convergence does not imply norm convergence

(A complete solution to part (c) should include remarks about how you know that if $\mathcal{H}$ is not finite dimensional, then there is an infinite orthonormal sequence in $\mathcal{H}$.)

Fall 2013 # 4. For each continuous function f on the interval $[0,1]$, define a function Tf by

$$(Tf)(x) = 1 + \lambda \int_0^x t f(t)\, dt.$$

a. Find a range of values for the parameter λ for which the transformation T is a contraction with respect to the supremum norm given by $\| g \|_{\infty} = \sup\{ |g(x)| \mid x \in [0,1]\}$. Justify your answer.

b. Describe an iteration process for solving the integral equation

$$f(x) = 1 + \lambda \int_0^x t f(t)\, dt$$

specifying the transformation to be iterated and explaining how this leads to a solution. State the conclusion offered by the Banach fixed point theorem (the contraction mapping principle).

c. With $f_0(x) = 0$ for all x as the starting function, compute the iterates, $f_1(x)$, $f_2(x)$ and $f_3(x)$.

Fall 2013 # 5. For each of the following decide if it is a vector space over $\mathbb{R}$. Give reasons for your answers. (You may assume that the set of all real valued functions on the interval $[-1,1]$ is a vector space with the operations $(f+g)(x) = f(x)+g(x)$ and $(\lambda f)(x) = \lambda f(x)$.)

a. $A = \{f : [-1,1] \to \mathbb{R} : \ f \text{ is an even function } \}$

b. $B = \{f : [-1,1] \to \mathbb{R} : \ f \text{ is differentiable and } f(0) + f'(0) = 1\}$

c. $C = \{f : [-1,1] \to \mathbb{R} : f(0) + \int_{-1}^{1} \sin(x) f(x)\, dx = 0\}$

Fall 2013 # 6. Suppose $\mathcal{E} = \{e_1, e_2, e_3, \dots\}$ is an orthonormal basis (complete orthonormal system) for a Hilbert space $\mathcal{H}$. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be numbers. For v in $\mathcal{H}$, put $T_n v = \sum_{k=1}^{n} \lambda_k \langle v, e_k \rangle e_{k+1}$

a. Show that T_n is a linear operator from $\mathcal{H}$ into $\mathcal{H}$.

b Show that T_n is a bounded linear operator.

c. Show that the operator norm of T_n is $\| T_n \| = \max(\{| \lambda_1 |, | \lambda_2 |, \dots, | \lambda_n |\})$.

Fall 2013 # 7. Let $\mathcal{V}$ be the space of all continuous real valued 2π-periodic functions on $\mathbb{R}$ for which f' exists and is continuous. For f and g in $\mathcal{V}$, put

$$[f,g] = \int_{-\pi}^{\pi} f'(t) g'(t)\, dt \qquad \text{and} \qquad [[f,g]] = f(0)g(0) + [f,g]$$

a. Show that $[\cdot,\cdot]$ is not an inner product on $\mathcal{V}$

b. Show that $[[\cdot,\cdot]]$ is an inner product on $\mathcal{V}$

You may assume that $\langle f, g \rangle = \int_{-\pi}^{\pi} f(t)g(t)\, dt$ does give an inner product on $\mathcal{V}$. .

End of Exam