California State University – Los Angeles
Department of Mathematics and Computer Science
Master's Degree Comprehensive Examination

Linear Analysis Fall 2000

Hoffman, Meyer*, Verona

Do **five** of the following eight problems. Each problem is worth 20 points. Please write in a fairly soft pencil (number 2) (or in ink if you wish) so that your work will duplicate well. There should be a supply available.

Exams are being graded anonymously, so put your name only where directed and follow any instructions concerning identification code numbers.

Notation: $\mathbb{C}$ denotes the set of complex numbers.

$\mathbb{R}$ denotes the set of real numbers.

$\mathrm{Re}(z)$ denotes the real part of the complex number z.

$\mathrm{Im}(z)$ denotes the imaginary part of the complex number z.

$\bar{z}$ denotes the complex conjugate of the complex number z.

$|z|$ denotes the absolute value of the complex number z

$\mathcal{C}([a,b])$ denotes the space of all continuous functions on the interval $[a,b]$. If there is need to specify the possible values, $\mathcal{C}([a,b],\mathbb{R})$ will denote the space of all continuous real valued functions on $[a,b]$ and $\mathcal{C}([a,b],\mathbb{C})$ the space of all continuous complex valued functions.

$L^2([a,b])$ denotes the space of all functions on the inteval $[a,b]$ such that $\int_a^b |f(x)|^2\, dx < \infty$

MISCELLANEOUS FACTS

$$\sin(a+b) = \sin a\cos b + \cos a\sin b \qquad \cos(a+b) = \cos a\cos b - \sin a\sin b$$

$$2\sin a\sin b = \cos(a-b) - \cos(a+b) \qquad 2\cos a\cos b = \cos(a-b) + \cos(a+b)$$

$$2\sin a\cos b = \sin(a+b) + \sin(a-b) \qquad 2\cos a\sin b = \sin(a+b) - \sin(a-b)$$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{1}{4a}\sin(2ax) \qquad \int \cos^2(ax)\,dx = \frac{x}{2} + \frac{1}{4a}\sin(2ax)$$

$$\int x\sin(ax)\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax) \qquad \int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$$

---

1. Which of the following are vector spaces over $\mathbb{R}$. For each, if it is a vector space, explain how you know. If it is not, explain why not.

a. The set of all differentiable functions $f : [0,1] \to \mathbb{R}$ such that $f(0) = 1$

b. The set of all integrable functions $f : [0,1] \to \mathbb{R}$ such that $\int_0^1 f(x)\,dx = 0$

c. The set of all polynomials of degree no more than 3 together with the zero polynomial.

d. The set of all polynomials of degree exactly 3.

---

2. Which of the following define a norm on the space $\mathbb{R}^2$? For each, if it is a norm, proive it. If it is not explain how you know it is not.

a. $\|(x,y)\|_a = 3|x| + 2|y|$

b. $\|(x,y)\|_b = 3x + 2y$

c. $\|(x,y)\|_c = x^2 + y^2$

d. $\|(x,y)\|_d = \int_0^1 |x + ty|\ dt$

---

3. Let $\mathcal{P}^n$ be the space of all polynomials with real coefficients of degree no more than n together with the zero polynomial considered as a subspace of the space $\mathcal{C}([0,1],\mathbb{R})$

a. Define what it means for a set of vectors to be linearly independent.

b. Show that the polynomials $p_0(x) = 1$, $p_1(x) = 1 + 2x$, $p_2(x) = 1 + 2x + 3x^2$ form a linearly independent set over $\mathbb{R}$ when considered as functions on the interval $[0,1]$.

c. Show that p_0, p_1,and p_2 span the space $\mathcal{P}^2$.

d. Find polynomials e_0 and e_1 which are an orthonormal basis for $\mathcal{P}^1$ with respect to the inner product $\langle p, q \rangle = \int_0^1 p(t)q(t)\,dt$.

---

4. Suppose ϕ, ϕ_1, ϕ_2, ϕ_3, $\ldots$ are continuous real valued functions on the interval $[a,b]$.

a. Define each of the following

(i) $\phi_n \to \phi$ pointwise on $[a,b]$

(ii) $\phi_n \to \phi$ uniformly on $[a,b]$

(ii) $\phi_n \to \phi$ in L^2-mean on $[a,b]$

b. Show that if $\phi_n \to \phi$ in L^2-mean on $[a,b]$, then

$$\int_a^b \phi_n(x)\,dx \to \int_a^b \phi(x)\,dx \qquad \text{as } n \to \infty.$$

c. Give an example to show that $\phi_n \to \phi$ pointwise on $[a,b]$ does not imply that

$$\int_a^b \phi_n(x)\,dx \to \int_a^b \phi(x)\,dx \qquad \text{as } n \to \infty.$$

d. Give an example to show that $\phi_n \to \psi$ pointwise on $[a,b]$ does not imply that ψ is continuous on $[a,b]$.

---

5. a. Find the Fourier series for the function $f(x) = \begin{cases} 0, & -\pi \le x < 0 \\ 1, & 0 \le x \le \pi \end{cases}$ with respect to the orthonormal family

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \sin nx, \frac{1}{\sqrt{\pi}} \cos nx : n = 1, 2, 3, \dots \right\}$$

b. Use the result of part **(a)** to show that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}.$$

---

6. For continuous f on the interval $[0, 1]$, let

$$(Tf)(x) = x + \lambda \int_0^x (x - t) f(t)\, dt.$$

a. Find a range of values of λ for which the transformation T is a contraction with respect to the supremum norm on $C([0, 1])$.

b. Explain, with an explicit statement of the formula to be iterated, how to use the contraction mapping principle to generate a sequence of approximations to a solution to the integral equation

(VIE)
$$f(x) = x + \lambda \int_0^x (x - t) f(t)\, dt.$$

Starting with $f_1(x) = 1$ for all x, compute the next two approximations, $f_2(x)$ and $f_3(x)$.

c. Show that if f is a solution to (VIE), then f is a solution to the initial value problem

$$f''(x) = \lambda f(x) \qquad ; \quad f(0) = 0 \text{ and } f'(0) = 1.$$

---

7. Consider the differential operator L defined for twice differentiable functions f on the interval $[1, 2]$ by $L = -\frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}\right)$.

a. Find a function $G(x, t)$ such that for a given function $h(x)$ on $[1, 2]$, a solution to the boundary value problem

(BVP) $\quad (Lf)(x) = h(x) \quad$ for $\quad 1 \le x \le 2 \qquad$ with $f(1) = 0$ and $f'(2) = 0$

is given by

$$f(x) = \int_1^2 G(x, t) h(t)\, dt = \int_1^x G(x, t) h(t)\, dt + \int_x^2 G(x, t) h(t)\, dt.$$

b. Use the result of part **(a)** to solve the systen (BVP) with $h(x) = x$.

---

8. For each continuous function f on the interval $[0,1]$, let

$$(Kf)(x) = \int_0^1 \sin(\pi x)\sin(\pi t)f(t)\,dt$$

a. Show that K is a bounded linear operator on $C([0,1])$ (or $L^2([0,1])$) with respect to the L^2-norm

b. Find all nonzero eigenvalues and the associated eigenvectors for the operator K.

c. Solve the integral equation

$$\text{(IE)} \qquad f(x) = 1 + \lambda \int_0^1 \sin(\pi x)\sin(\pi t)f(t)\,dt$$

End of Exam