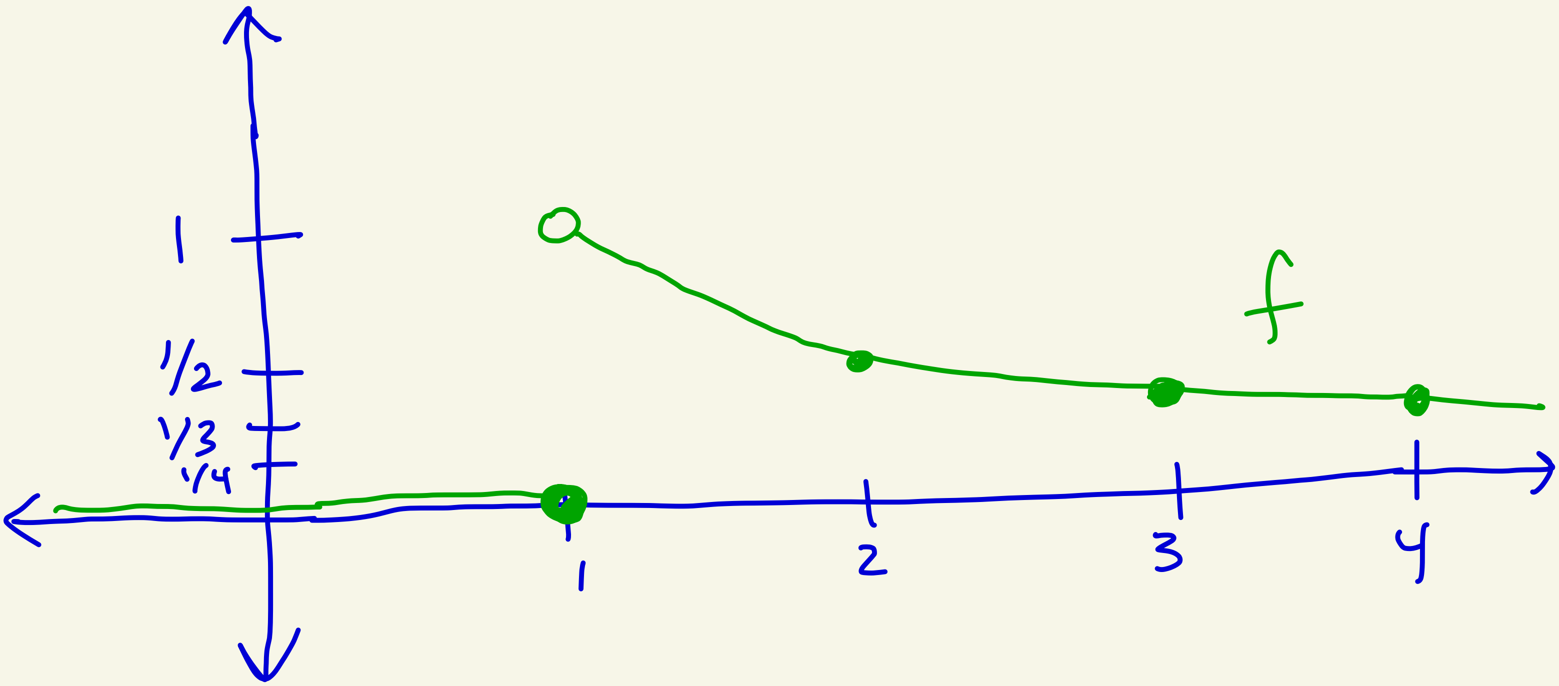
Math 5800
HW 8 Solutions

More on Integrable Functions

① Let

$$f(x) = \begin{cases} 0 & \text{if } x \le 1 \\ \frac{1}{x} & \text{if } x > 1 \end{cases}$$



Consider the step function

$$\varphi_n = \sum_{k=2}^{n} \frac{1}{k} \cdot \chi_{(k-1,k]}$$

So,

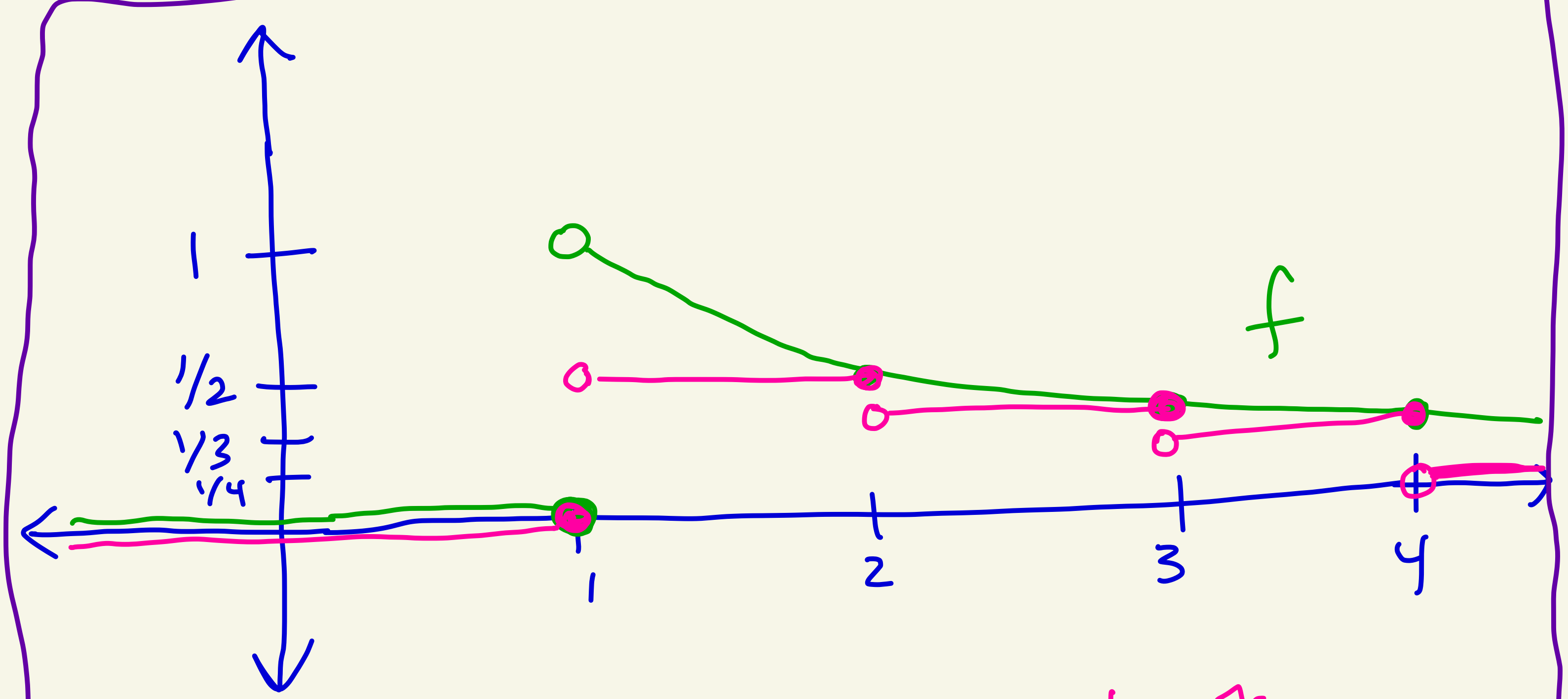
$$\varphi_1 = \frac{1}{2} \cdot \chi_{(1,2]}$$

$$\varphi_2 = \frac{1}{2} \cdot \chi_{(1,2]} + \frac{1}{3} \cdot \chi_{(2,3]}$$

etc.

Note that $\varphi_n(x) \leq f(x)$ for all $x \in \mathbb{R}$.

For example here is a picture with φ_3 and f on the same graph.



$$\varphi_3 = \frac{1}{2} \cdot \chi_{(1,2]} + \frac{1}{3} \cdot \chi_{(2,3]} + \frac{1}{4} \cdot \chi_{(3,4]}$$

Since φ_n is a step function for each $n \geq 1$ we have that $\varphi_n \in L^1$ for each $n \geq 1$.

We now show that $f \notin L^1$.

Suppose instead that $f \in L^1$.

Then $\int f$ is a finite real number.

For each $n \geq 1$, since $\varphi_n \in L^1$ and $f \in L^1$ and $\varphi_n \leq f$ we have that $\int \varphi_n \leq \int f$.

Also, if $n \geq 1$, then

$$\int \varphi_n = \sum_{k=2}^{n} \frac{1}{k} \cdot \ell\big((k-1, k]\big) = \sum_{k=2}^{n} \frac{1}{k}$$

Thus, for all $n \geq 1$ we have that

$$\underbrace{\sum_{k=2}^{n} \frac{1}{k}}_{\frac{1}{2} + \cdots + \frac{1}{n}} \leq \int f.$$

But, $\lim_{n \to \infty} \sum_{k=2}^{n} \frac{1}{k} = \infty$

because it is $\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$

which is one off from

the harmonic series.

This implies that $\int f$ is not

a finite real number.

Contradiction.

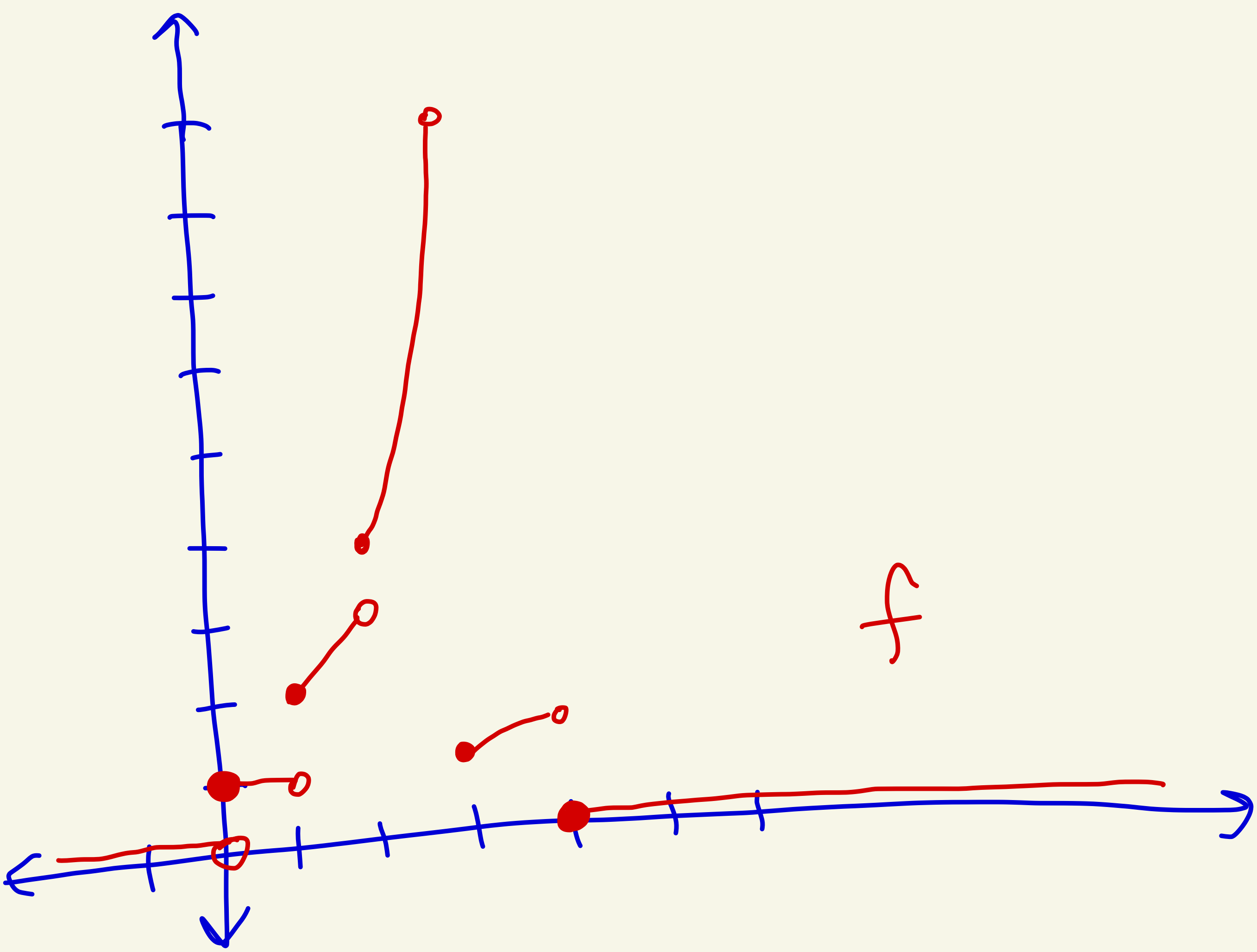
Thus, $f \notin L^1$.

So, f is not Lebesgue

integrable. $\blacksquare$

② Let

$$f(x) = \begin{cases} 1 & \text{if } x \in [0,1) \\ x+1 & \text{if } x \in [1,2) \\ x^2 & \text{if } x \in [2,3) \\ \ln(x) & \text{if } x \in [3,4) \\ 0 & \text{otherwise} \end{cases}$$



[Note that $\ln(3) \approx 1.0986$, $\ln(4) \approx 1.3863$]

f is bounded on $[0,4]$.

f vanishes outside $[0,4]$.

And

$E = \{ x \in (0,4) \mid f \text{ is discontinuous at } x \}$

$= \{1, 2, 3\}$

has measure zero.

Thus, from a theorem in class

$f \in L^1$.

So, f is Lebesgue integrable.

Since $f = f \cdot \chi_{[0,4)}$

everywhere and $f \in L^1$ we

have that $f \cdot \chi_{[0,4)} \in L^1$.

Thus, $f \in L^1([0,4))$.

$\blacksquare$