Math 5800 Homework #6

Sequences of functions and the standard construction

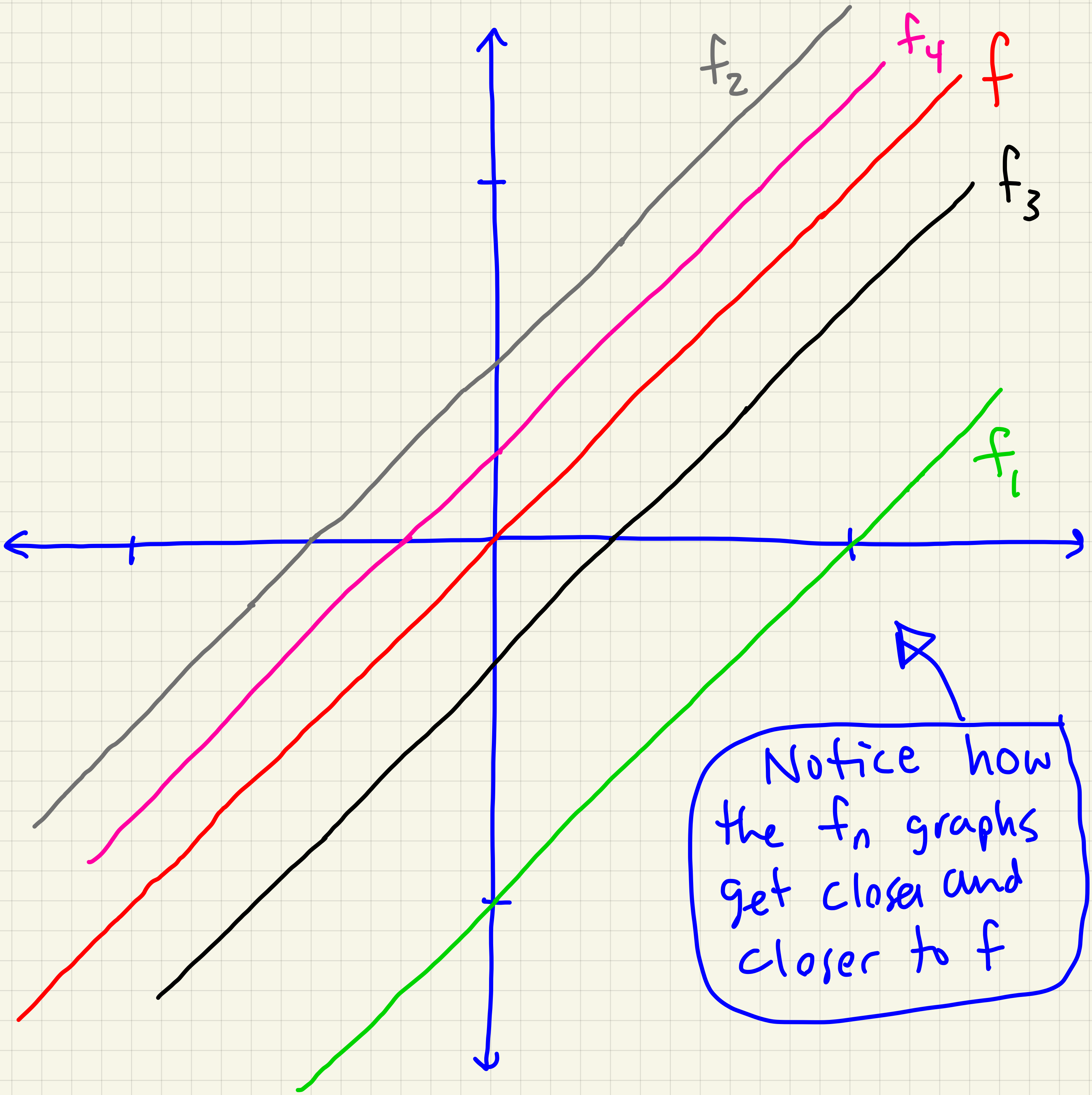
① (a) $f(x) = x$

$f_1(x) = x - 1$

$f_3(x) = x - \frac{1}{3}$

$f_2(x) = x + \frac{1}{2}$

$f_4(x) = x + \frac{1}{4}$



Notice how the f_n graphs get closer and closer to f

① (b) $f(1) = 1$

$f_1(1) = 1 - 1 = 0$

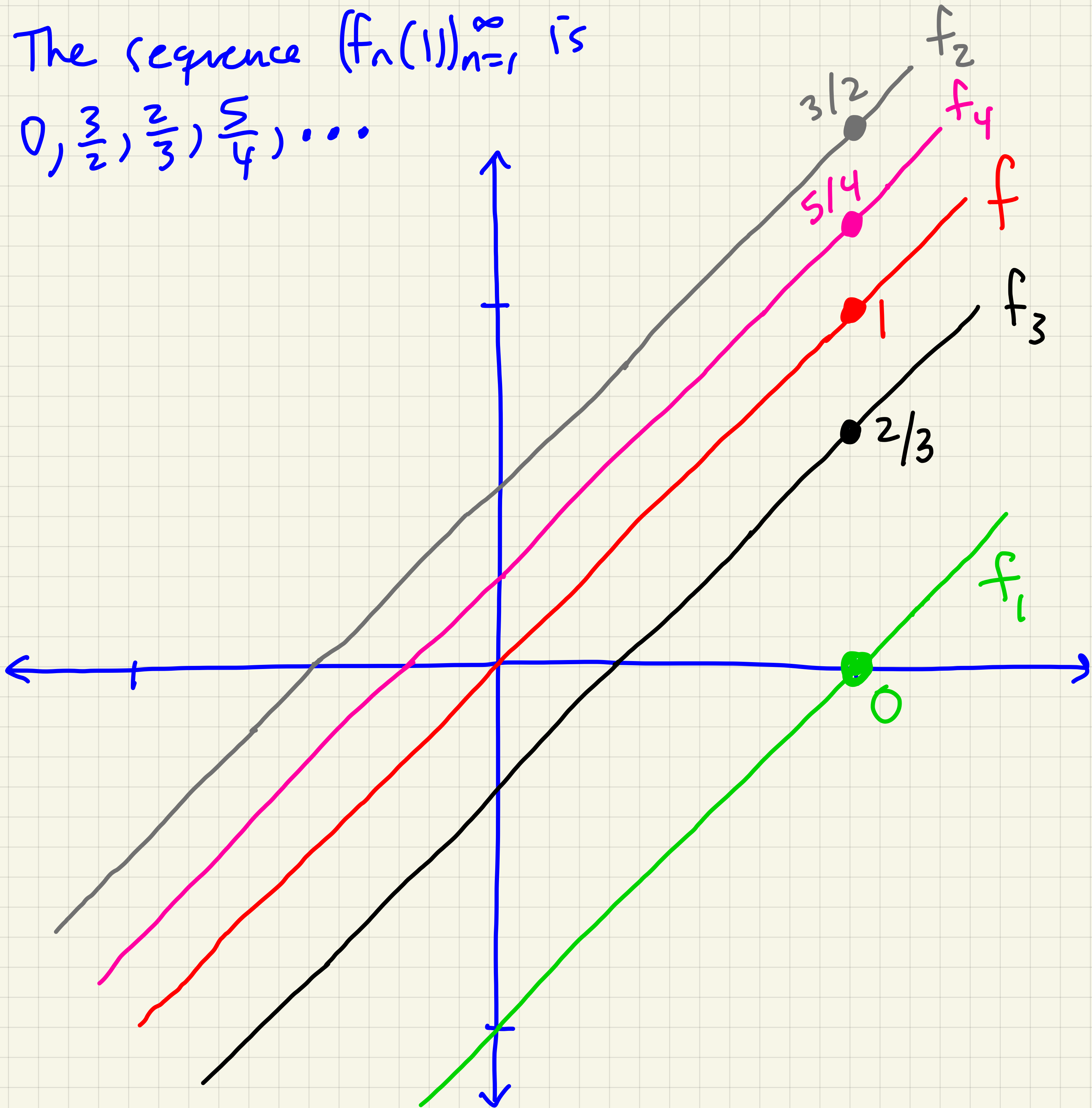
$f_2(1) = 1 + \frac{1}{2} = \frac{3}{2}$

$f_3(1) = 1 - \frac{1}{3} = \frac{2}{3}$

$f_4(1) = 1 + \frac{1}{4} = \frac{5}{4}$

The sequence $(f_n(1))_{n=1}^{\infty}$ is

$0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \ldots$



① (c) For any $n \geq 1$ we have that

$$f_n(1) = 1 + \frac{(-1)^n}{n}$$

Note that

$$\boxed{-\frac{1}{n} \leq \frac{(-1)^n}{n} \leq \frac{1}{n}}$$

for any $n \geq 1$.

Since $\lim\limits_{n\to\infty} -\frac{1}{n} = -\lim\limits_{n\to\infty} \frac{1}{n} = -0 = 0$

and $\lim\limits_{n\to\infty} \frac{1}{n} = \lim\limits_{n\to\infty} \frac{1}{n} = -0 = 0$,

by the squeeze theorem [4650 HW 2] we have that $\lim\limits_{n\to\infty} \frac{(-1)^n}{n} = 0$.

Thus, $\lim\limits_{n\to\infty} f_n(1) = \lim\limits_{n\to\infty} \left[1 + \frac{(-1)^n}{n}\right]$

$= \lim\limits_{n\to\infty} 1 + \lim\limits_{n\to\infty} \frac{(-1)^n}{n} = 1 + 0 = 1.$

1 (d) Let $x \in \mathbb{R}$ be fixed.
From 1c we know that $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0.$

Thus,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left[x + \frac{(-1)^n}{n} \right]$$

$$= \lim_{n\to\infty} x + \lim_{n\to\infty} \frac{(-1)^n}{n}$$

$$= x + 0$$

$$= x = f(x)$$

Thus, $f_n(x) \to f(x)$ for all $x \in \mathbb{R}$.

Said another way, $f_n \to f$ on all of $\mathbb{R}$.

② (a)

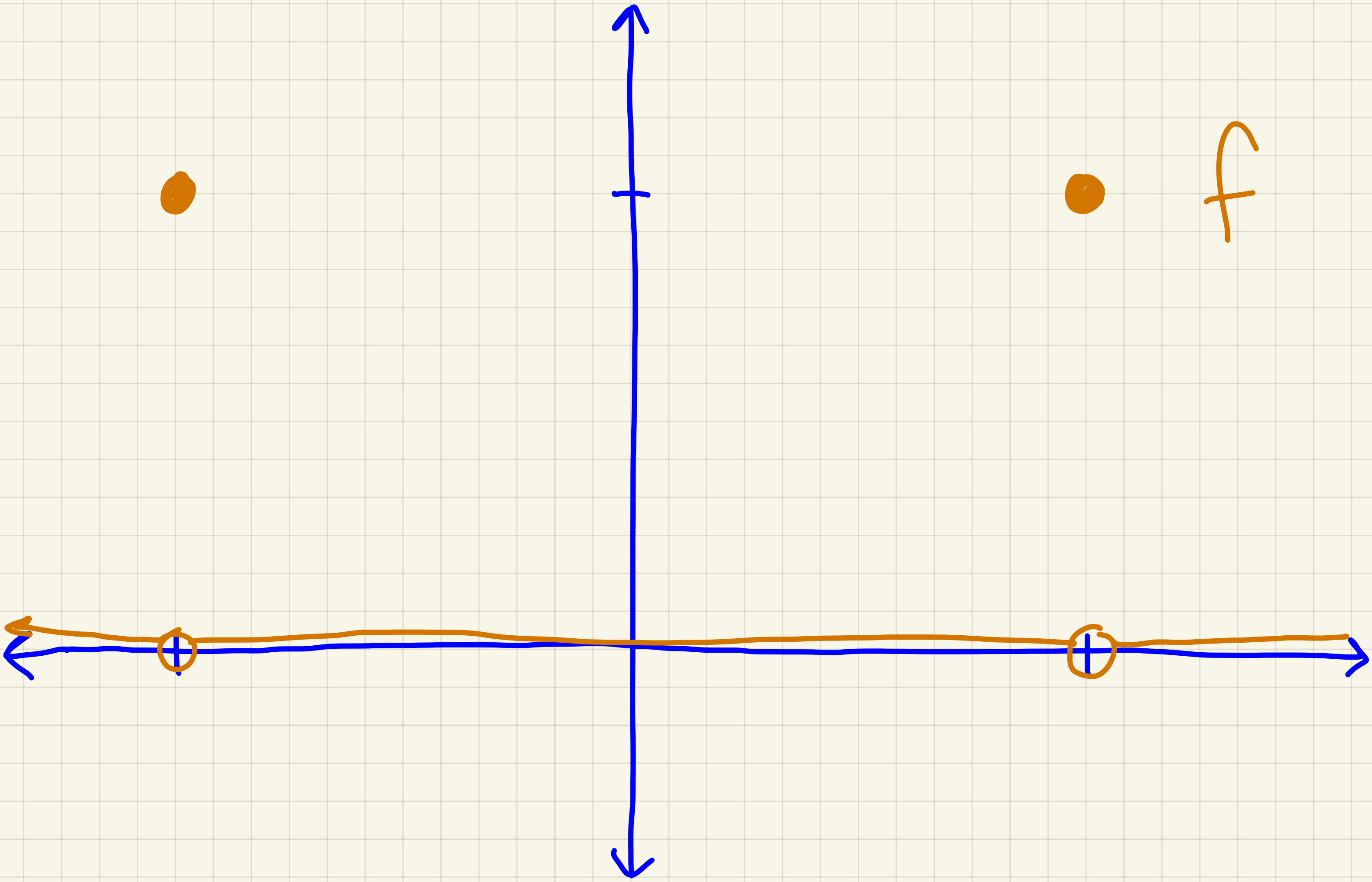
$$f(x) = \begin{cases} 1 & \text{if } x = 1 \text{ or } x = -1 \\ 0 & \text{otherwise} \end{cases}$$

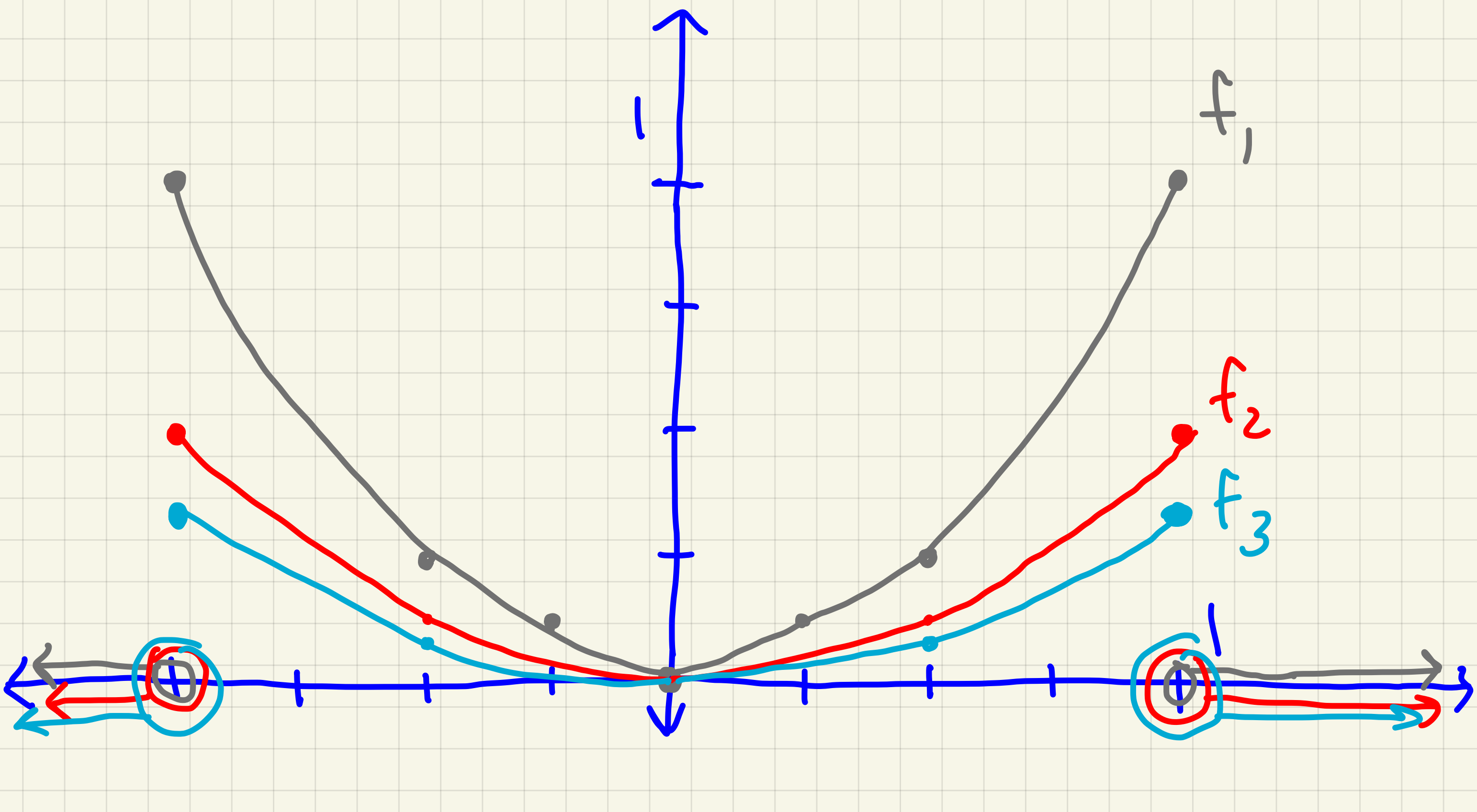
$$f_1(x) = \begin{cases} x^2, & \text{if } x \in [-1, 1] \\ 0, & \text{otherwise} \end{cases}$$

$$f_3(x) = \begin{cases} \frac{x^2}{3}, & \text{if } x \in [-1, 1] \\ 0, & \text{otherwise} \end{cases}$$

$$f_2(x) = \begin{cases} \frac{x^2}{2}, & \text{if } x \in [-1, 1] \\ 0, & \text{otherwise} \end{cases}$$

$$f_4(x) = \begin{cases} \frac{x^2}{4}, & \text{if } x \in [-1, 1] \\ 0, & \text{otherwise} \end{cases}$$


f


1
f1
f2
f3
1

② (b) Let $x \in \mathbb{R} - \{1, -1\}$.

Then,

$$\lim_{n\to\infty} \frac{x^2}{n} = x^2 \cdot \lim_{n\to\infty} \frac{1}{n}$$

$$= x^2 \cdot 0 = 0$$

Since

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \text{ or } x = -1 \\ 0 & \text{otherwise} \end{cases}$$

we have that

$$\lim_{n\to\infty} f_n(x) = 0 = f(x)$$

So, $f_n \to f$ on $\mathbb{R} - \{1, -1\}$

2 (c)

If $x = 1$ or $x = -1$, then

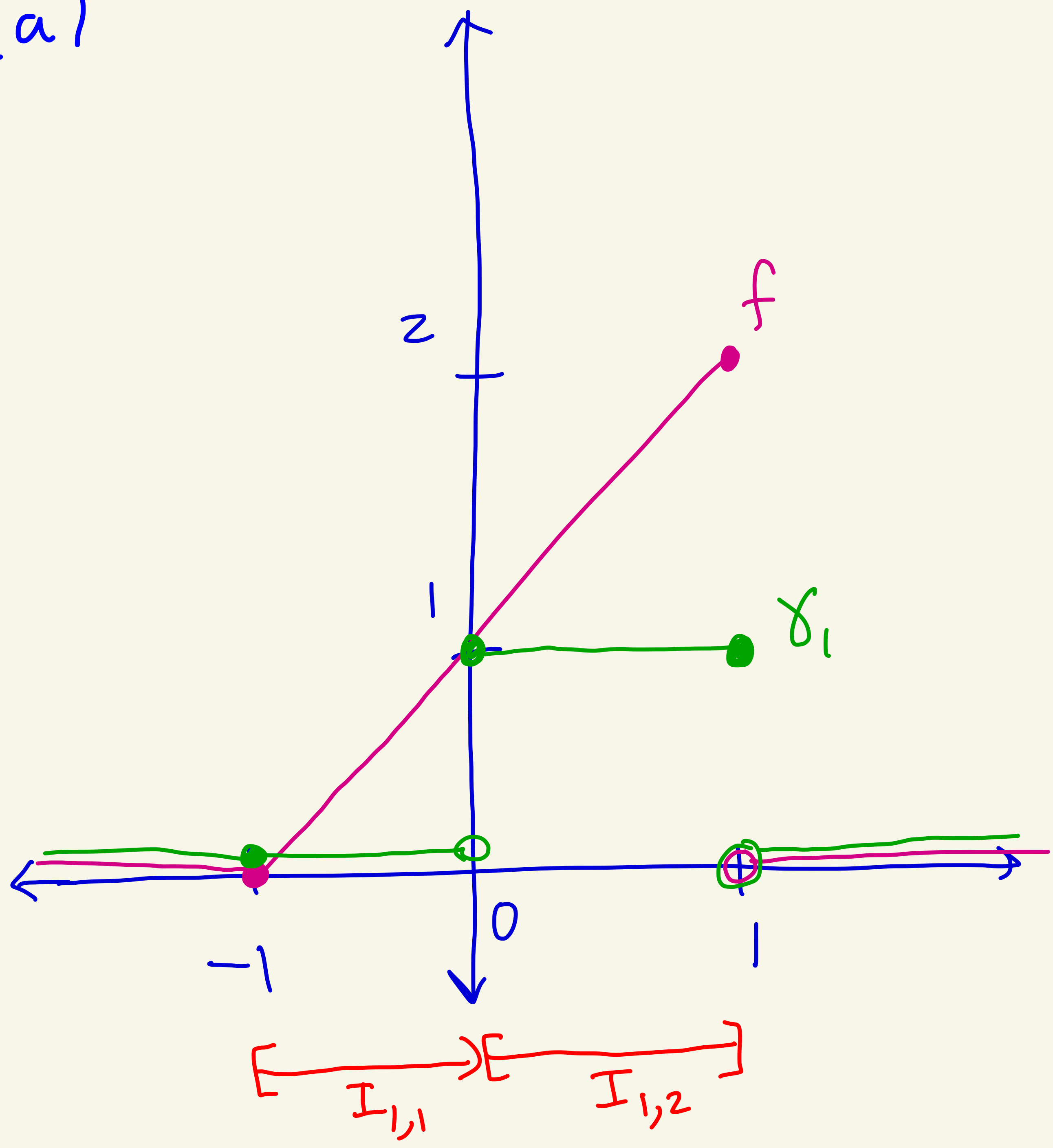
$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{(\pm 1)^2}{n} = \lim_{n\to\infty} \frac{1}{n} = 0$$

but $f(x) = 1$.

Thus, $\lim_{n\to\infty} f_n(x) \neq f(x)$

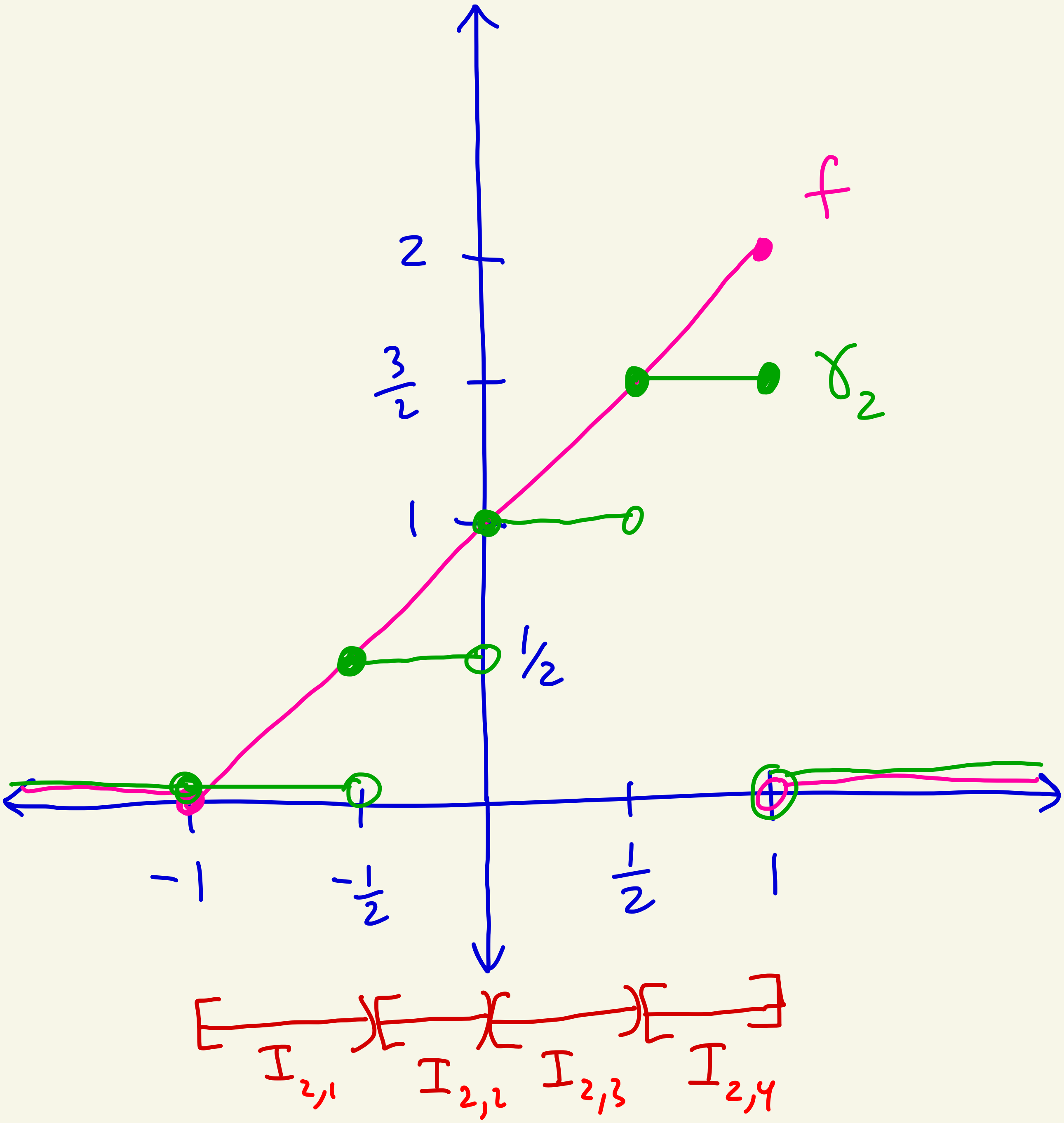
if $x \in \{1, -1\}$.

③ (a)



$$\gamma_1 = 0 \cdot \chi_{[-1,0)} + 1 \cdot \chi_{[0,1]}$$

$$= \chi_{[0,1]}$$



$$\varphi_2 = 0 \cdot \chi_{[-1,-\frac{1}{2})} + \frac{1}{2} \cdot \chi_{[-1/2,\,0)} + 1 \cdot \chi_{[0,\frac{1}{2})} + \frac{3}{2} \cdot \chi_{[\frac{1}{2},1]}$$

$$\varphi_2 = \frac{1}{2} \cdot \chi_{[-1/2,\,0)} + 1 \cdot \chi_{[0,\frac{1}{2})} + \frac{3}{2} \cdot \chi_{[\frac{1}{2},1]}$$

③ (b) Claim 1: If $x \in [-1,1]$, then, $|\gamma_n(x) - f(x)| \leq \frac{1}{2^{n-1}}$

pf of claim 1: Let $n \geq 1$ be fixed. Let $x \in [-1,1)$. We will deal with $x = 1$ after this. Then, $\Delta_n = \frac{b-a}{2^n} = \frac{1-(-1)}{2^n} = \frac{1}{2^{n-1}}$

We have $x \in [-1+(k-1)\Delta_n, -1+k\cdot\Delta_n)$

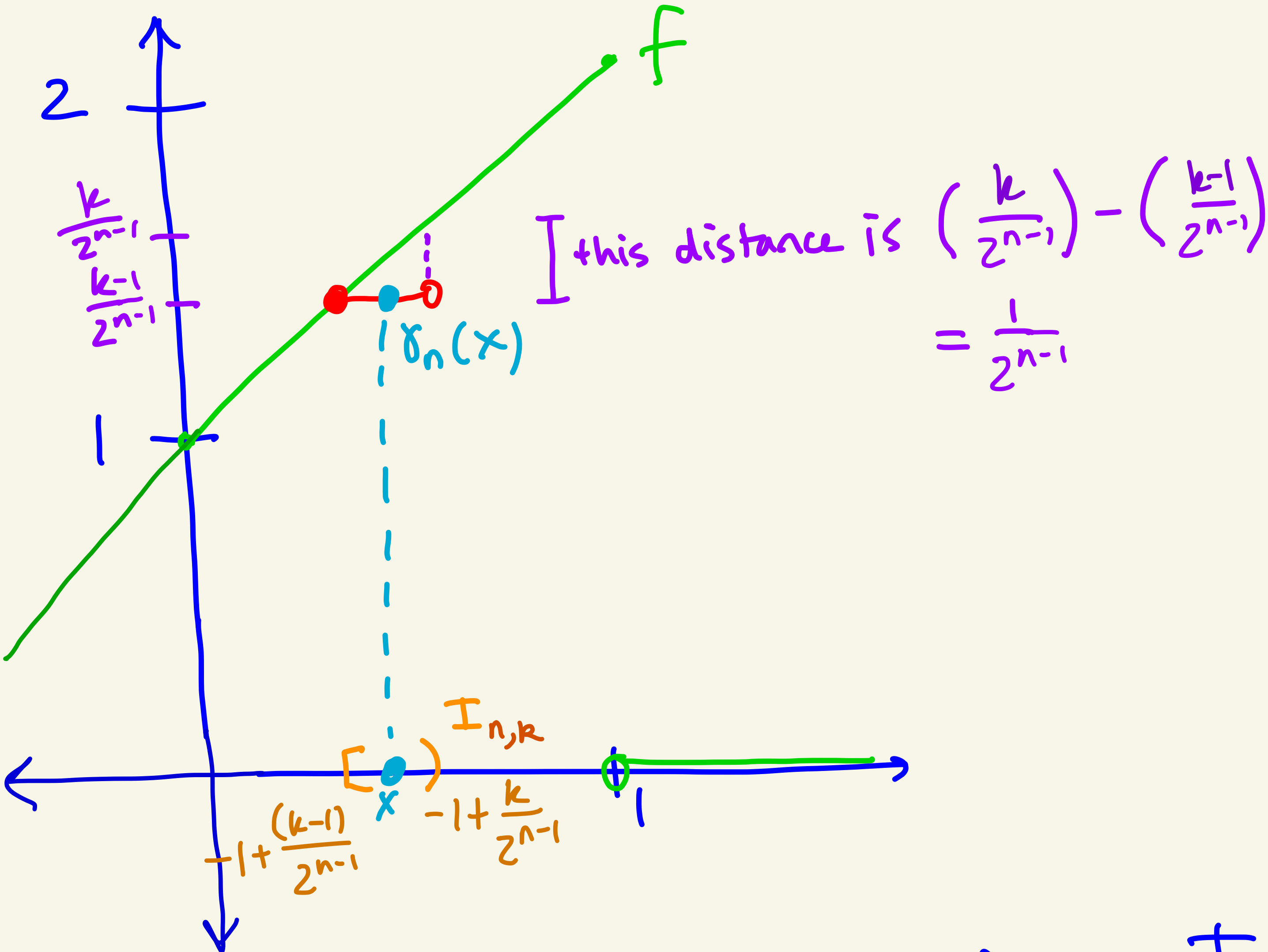
$$= \left[-1+\frac{k-1}{2^{n-1}}, -1+\frac{k}{2^{n-1}}\right)$$

for some $1 \leq k \leq 2^n$.

Since $f(x) = x+1$ is an increasing function,

$$\gamma_n(x) = \inf\{f(t) \mid t \in I_{n,k}\}$$
$$= f\left(-1+\frac{k-1}{2^{n-1}}\right) = -1+\frac{k-1}{2^{n-1}}+1 = \frac{k-1}{2^{n-1}}$$

Note the following diagram.



From above we see that if $x \in I_{n,k}$

then $\left|\gamma_n(x) - f(x)\right| < \frac{1}{2^{n-1}}$

What if $x = 1$? Then,

$$\gamma_n(1) = \inf\left\{ f(t) \;\middle|\; -1 + \frac{2^n - 1}{2^{n-1}} \le t \le -1 + \frac{2^n}{2^{n-1}} \right\}$$

$$= f\left(-1 + \frac{2^n - 1}{2^{n-1}}\right) = 1 + \left(-1 + \frac{2^n - 1}{2^{n-1}}\right) = \frac{2^n - 1}{2^{n-1}}$$

So,

$$\left|\gamma_n(1) - f(1)\right| = \left|\left(\frac{2^n-1}{2^{n-1}}\right) - 2\right|$$

$$= \left|\frac{2^n-1}{2^{n-1}} - 2\right| = \left|\frac{2^n - 1 - 2^n}{2^{n-1}}\right|$$

$$= \left|\frac{-1}{2^{n-1}}\right| = \frac{1}{2^{n-1}}$$

$\boxed{\text{claim 1}}$

<u>Claim 2:</u> If $x \in [-1,1]$, then $\lim_{n\to\infty} \gamma_n(x) = f(x)$.

<u>pf of claim 2:</u> Let $\varepsilon > 0$ and $-1 \leq x \leq 1$.

Then $\left|\gamma_n(x) - f(x)\right| \leq \frac{1}{2^{n-1}}$

We want to make $\frac{1}{2^{n-1}} < \varepsilon$.

Note that $\frac{1}{2^{n-1}} < \varepsilon$ iff $\frac{1}{\varepsilon} < 2^{n-1}$

iff $\log_2\left(\frac{1}{\varepsilon}\right) < n-1$

iff $\log_2\left(\frac{1}{\varepsilon}\right) + 1 < n$.

Set $N > \log_2\left(\frac{1}{\varepsilon}\right) + 1$

Then if $n \geq N > \log_2\left(\frac{1}{\varepsilon}\right) + 1$

we have that

$$|\gamma_n(x) - f(x)| \leq \frac{1}{2^{n-1}} < \varepsilon$$

if $x \in [-1, 1]$.

So, $\lim_{n \to \infty} \gamma_n(x) = f(x)$

$\boxed{\text{Claim 2}}$

$\blacksquare$

③ (c) We know that $\gamma_n \to f$ on $[-1,1]$.

If $x \notin [-1,1]$, then $\gamma_n(x) = 0 = f(x)$.

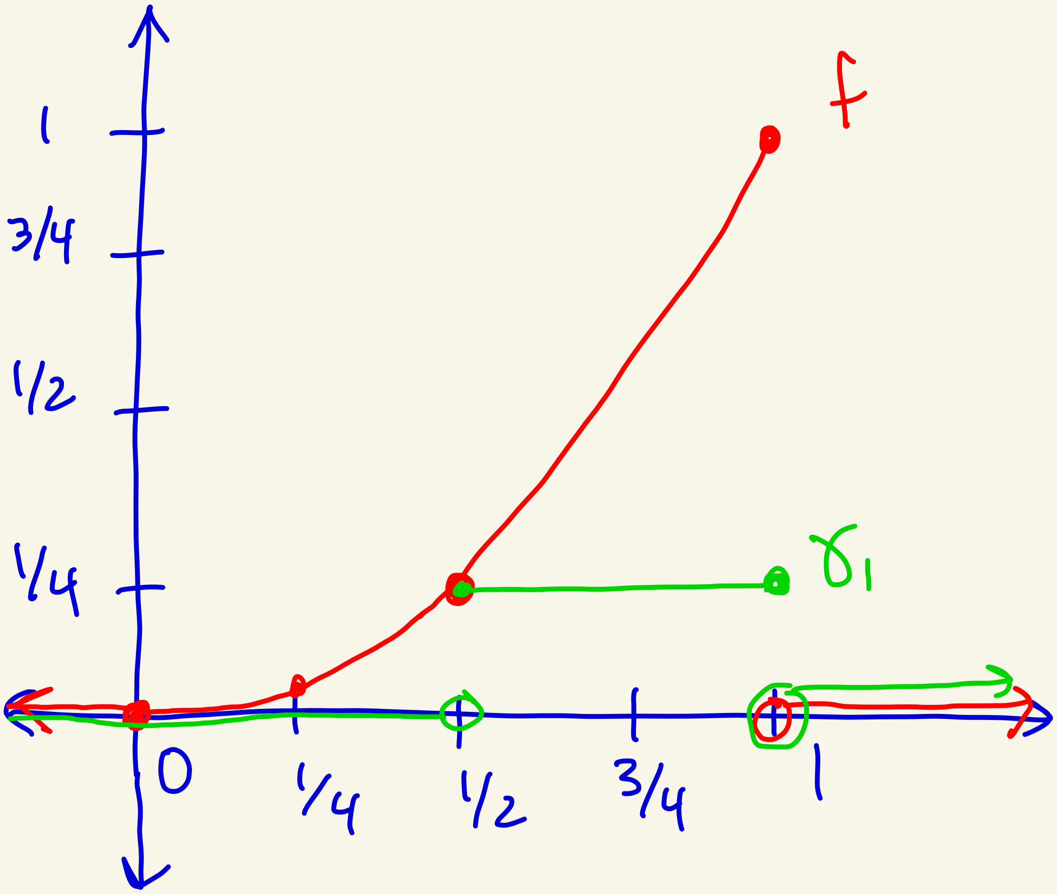
Thus, if $x \notin [-1,1]$, then

$$\lim_{n\to\infty} \gamma_n(x) = \lim_{n\to\infty} 0 = 0 = f(x).$$

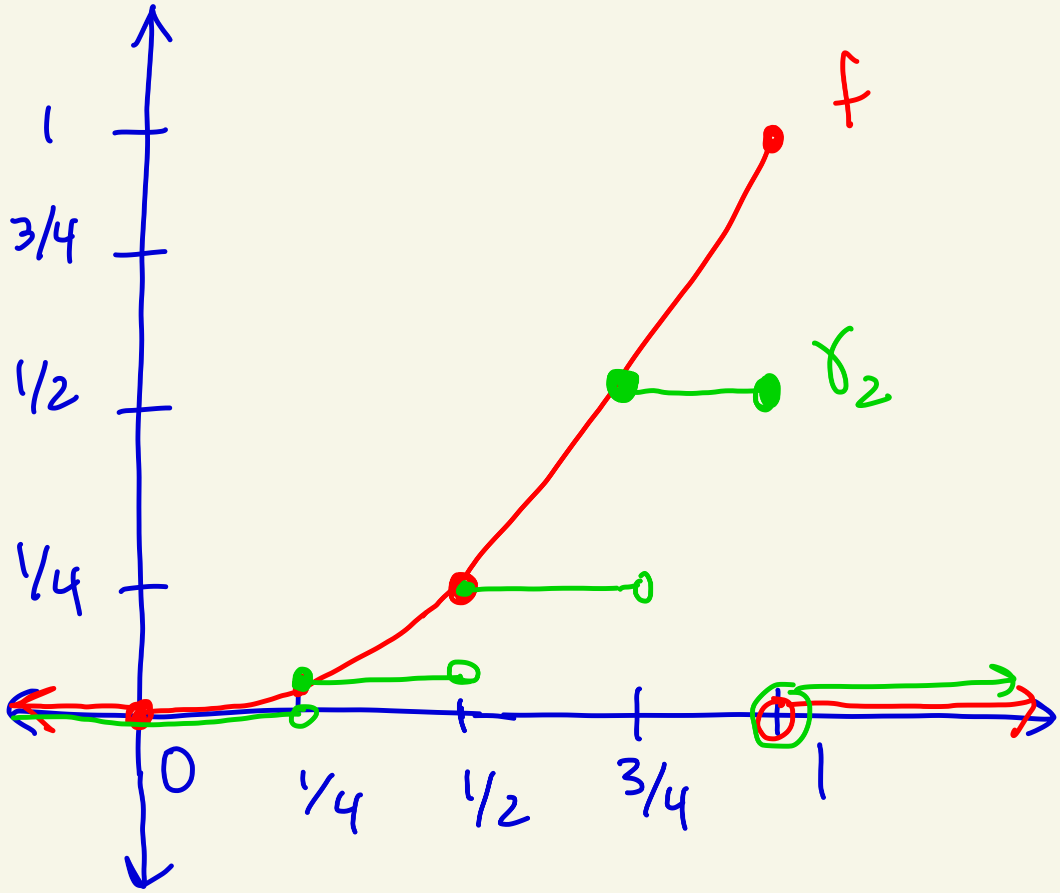
So, $\gamma_n \to f$ on $\mathbb{R} - [-1,1]$ also.

Thus, $\gamma_n \to f$ on all of $\mathbb{R}$.

④ (a)



$$\gamma_1 = 0 \cdot \chi_{[0,\frac{1}{2})} + \frac{1}{4} \cdot \chi_{[\frac{1}{2},1]}$$



$$\gamma_2 = 0 \cdot \chi_{[0,\frac{1}{4})} + \left(\frac{1}{4}\right)^2 \cdot \chi_{[\frac{1}{4},\frac{1}{2})} + \left(\frac{1}{2}\right)^2 \cdot \chi_{[\frac{1}{2},\frac{3}{4})} + \left(\frac{3}{4}\right)^2 \cdot \chi_{[\frac{3}{4},1]}$$

$$= \frac{1}{16} \cdot \chi_{[\frac{1}{4},\frac{1}{2})} + \frac{1}{4} \cdot \chi_{[\frac{1}{2},\frac{3}{4})} + \frac{9}{16} \cdot \chi_{[\frac{3}{4},1]}$$

4 (b)

Claim 1: If $x \in [0,1]$, then $|f(x) - \gamma_n(x)| < \frac{1}{2^{n-1}}$

Let $n \geq 1$ be fixed.

Let $x \in [0,1)$.

We will deal with $x=1$ at the end.

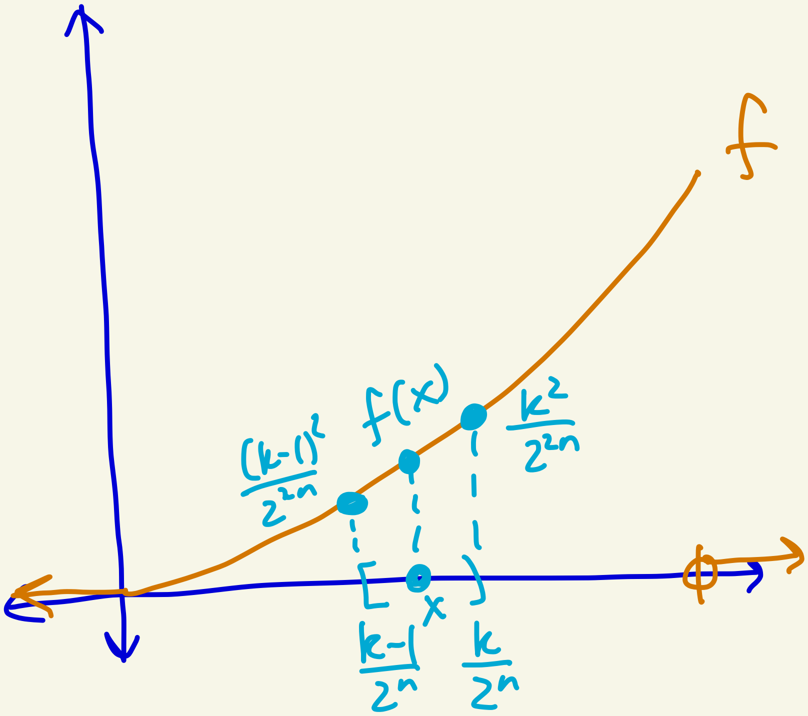
Then, $x \in \left[0 + (k-1)\frac{1}{2^n}, 0 + k \cdot \frac{1}{2^n}\right) = \left[\frac{(k-1)}{2^n}, \frac{k}{2^n}\right)$

for some $1 \leq k \leq 2^n$.

Since f is an increasing function on $[0,1]$ we have that

$$\gamma_n(x) = \inf\left\{f(t) \,\middle|\, \frac{k-1}{2^n} \leq t < \frac{k}{2^n}\right\}$$

$$= f\left(\frac{k-1}{2^n}\right) = \left(\frac{k-1}{2^n}\right)^2 = \frac{k^2-2k+1}{2^{2n}}$$



Then, $|\gamma_n(x) - f(x)| = |f(x) - \gamma_n(x)|$

$$< \underbrace{\frac{k^2}{2^{2n}}}_{\substack{\text{largest } f(x) \text{ gets} \\ \text{is at } \frac{k}{2^n} \text{ and } x < \frac{k}{2^n}}} - \underbrace{\frac{k^2 - 2k + 1}{2^{2n}}}_{\gamma_n(x)}$$

$$= \frac{2k-1}{2^{2n}} \underset{1 \le k \le 2^n}{\le} \frac{2 \cdot 2^n - 1}{2^n \cdot 2^n} = \frac{2 \cdot 2^n}{2^n \cdot 2^n} - \frac{1}{2^n \cdot 2^n}$$

$$= \frac{1}{2^{n-1}} - \frac{1}{2^n \cdot 2^n} < \frac{1}{2^{n-1}}.$$

Now if $x=1$, then $f(1)=1^2=1$

and $\gamma_n(1) = \left[(2^n-1)\cdot\frac{1}{2^n}\right]^2$

So, $|f(x)-\gamma_n(x)| = 1-\left(\frac{2^n-1}{2^n}\right)^2$

$\boxed{f(x) > \gamma_n(x)}$

$$= \frac{2^{2n}}{2^{2n}} - \frac{2^{2n}-2\cdot 2^n+1}{2^{2n}} = \frac{2\cdot 2^n - 1}{2^n\cdot 2^n}$$

$$< \frac{2\cdot 2^n}{2^n\cdot 2^n} = \frac{2}{2^n} = \frac{1}{2^{n-1}}$$

$\boxed{\text{Claim 1}}$

Claim 2: $\gamma_n \to f$ pointwise on $[0,1]$

Let $x \in [0,1]$.

Let $\varepsilon > 0$.

From claim 1, we have that

$$|\gamma_n(x) - f(x)| < \frac{1}{2^{n-1}}.$$

Note that $\frac{1}{2^{n-1}} < \varepsilon$ iff $\frac{1}{\varepsilon} < 2^{n-1}$

iff $\log_2\left(\frac{1}{\varepsilon}\right) < n-1$

iff $\log_2\left(\frac{1}{\varepsilon}\right) + 1 < n$.

Set $N > \log_2\left(\frac{1}{\varepsilon}\right) + 1$.

Then if $n \geq N$ we have that

$$|\gamma_n(x) - f(x)| < \varepsilon.$$

So, $\gamma_n \to f$ on $[0,1]$.

$\boxed{\text{claim 2}}$ $\blacksquare$

④ (c)

From 4b, we have that $\lim_{n\to\infty} \gamma_n(x) = f(x)$ if $x \in [0,1]$.

If $x \notin [0,1]$, then $\lim_{n\to\infty} \gamma_n(x) = \lim_{n\to\infty} 0 = 0 = f(x)$.

Hence $\gamma_n \to f$ on all of $\mathbb{R}$. $\blacksquare$

5 Let $x \in A$ be fixed.

Since $f_n \to f$ on A we know that

$$\lim_{n\to\infty} f_n(x) = f(x).$$

Since $g_n \to g$ on A we know that

$$\lim_{n\to\infty} g_n(x) = g(x).$$

Thus, from 4650 HW 2,

$$\lim_{n\to\infty} \left[f_n(x) + g_n(x)\right]$$

$$= \lim_{n\to\infty} f_n(x) + \lim_{n\to\infty} g_n(x)$$

$$= f(x) + g(x)$$

So, $f_n + g_n \longrightarrow f + g$ on A.

6

Since $f_n \to f$ almost everywhere on $\mathbb{R}$ there exists $A_1 \subseteq \mathbb{R}$ with $f_n(x) \to f(x)$ for all $x \in A_1$ and $\mathbb{R} - A_1$ has measure zero.

Since $g_n \to g$ almost everywhere on $\mathbb{R}$ there exists $A_1 \subseteq \mathbb{R}$ with $g_n(x) \to g(x)$ for all $x \in A_2$ and $\mathbb{R} - A_2$ has measure zero.

Let $A = A_1 \cap A_2$.

Then, $\mathbb{R} - A = (\mathbb{R} - A_1) \cup (\mathbb{R} - A_2)$. and so since the union of two sets of measure zero is measure zero we know that $\mathbb{R} - A$ has measure zero.

Let $x \in A_1 \cap A_2$ be fixed.

Since $x \in A_1$, $f_n(x) \to f(x)$ as $n \to \infty$

Since $x \in A_2$, $g_n(x) \to g(x)$ as $n \to \infty$

So, $f_n(x) + g_n(x) \to f(x) + g(x)$ as $n \to \infty$

Thus, $f_n + g_n \to f + g$ on A where A is an almost everywhere set.

(since $\mathbb{R} - A$ has measure zero).

So, $f_n + g_n$ converges to $f + g$ almost everywhere. $\blacksquare$