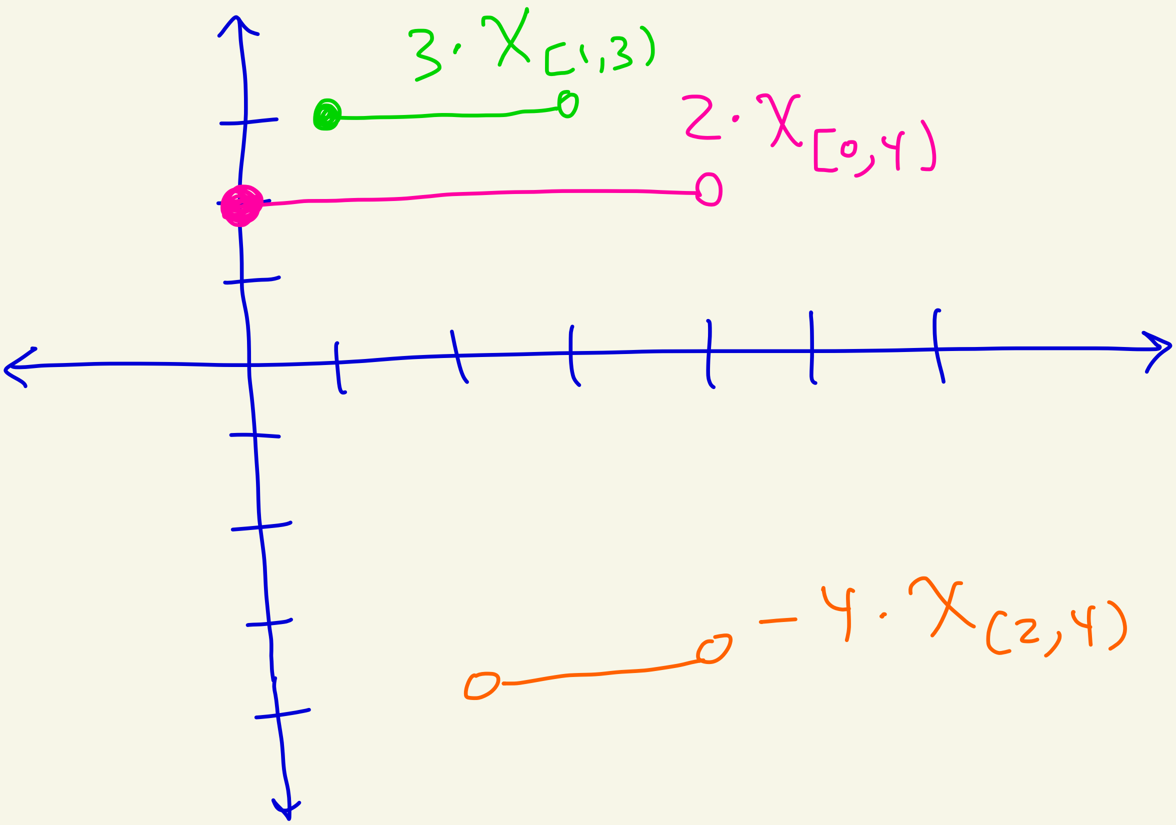
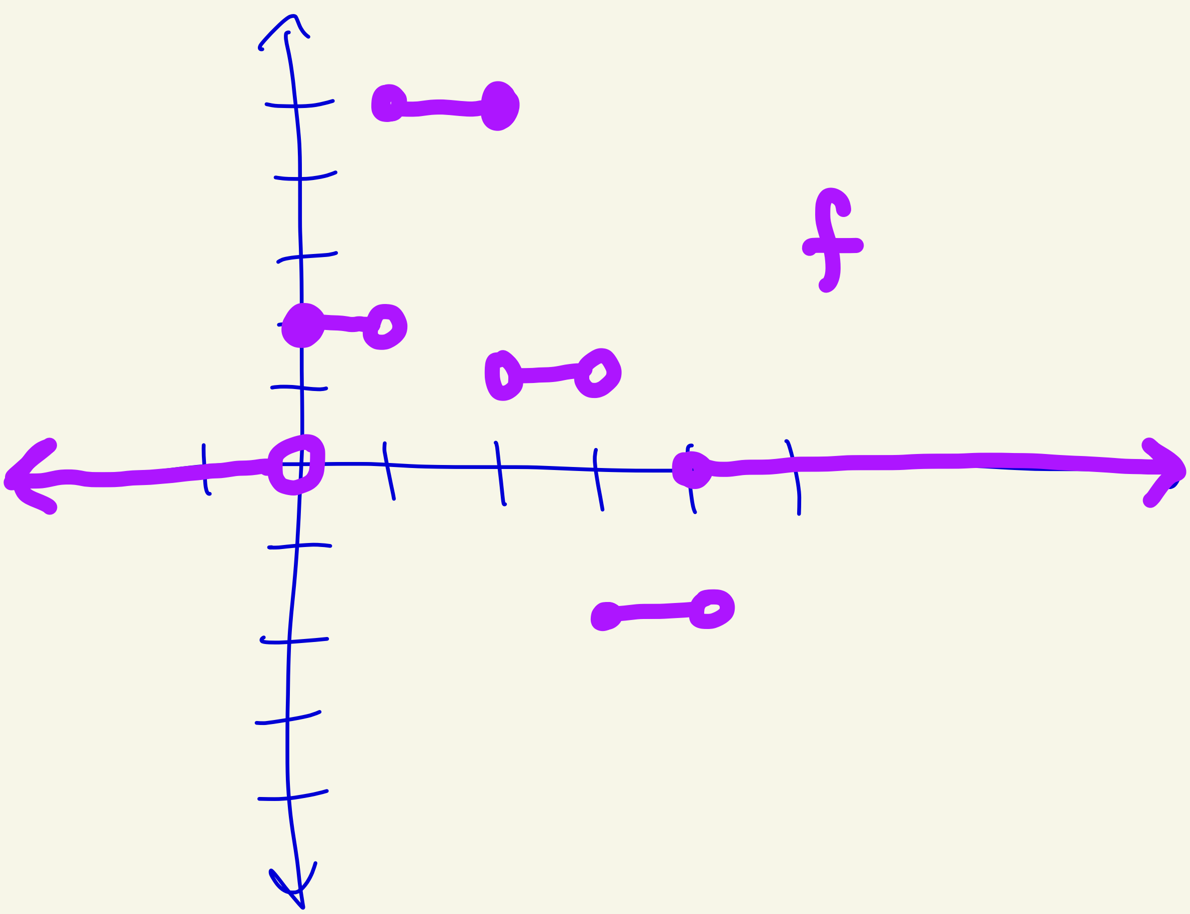
Homework #4
Solutions

Step Functions

① (a) $f = 2\cdot\chi_{[0,4)} + 3\cdot\chi_{[1,3)} - 4\cdot\chi_{(2,4)}$



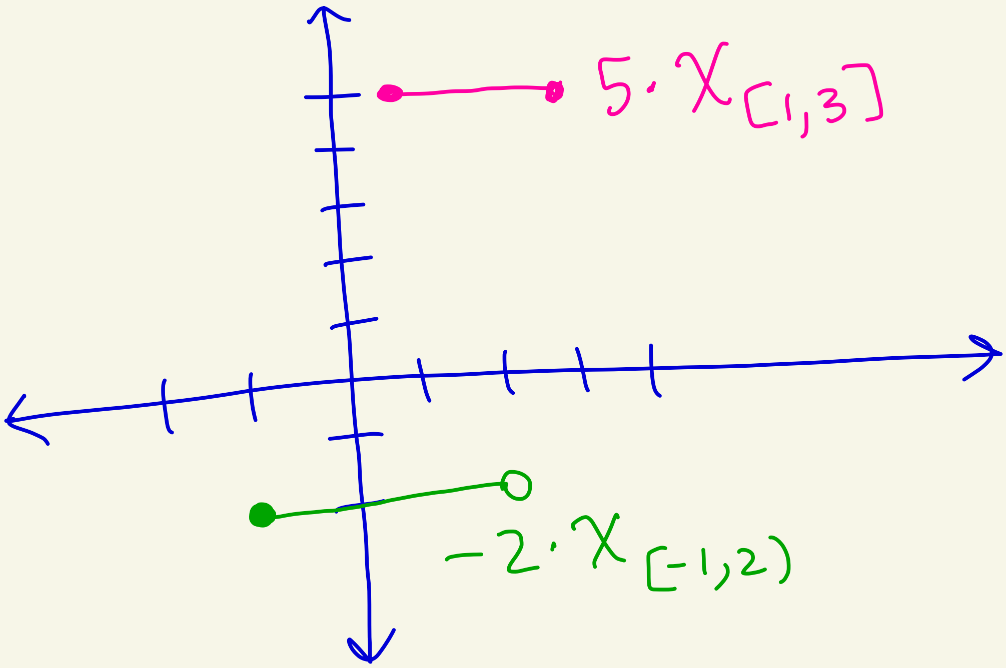


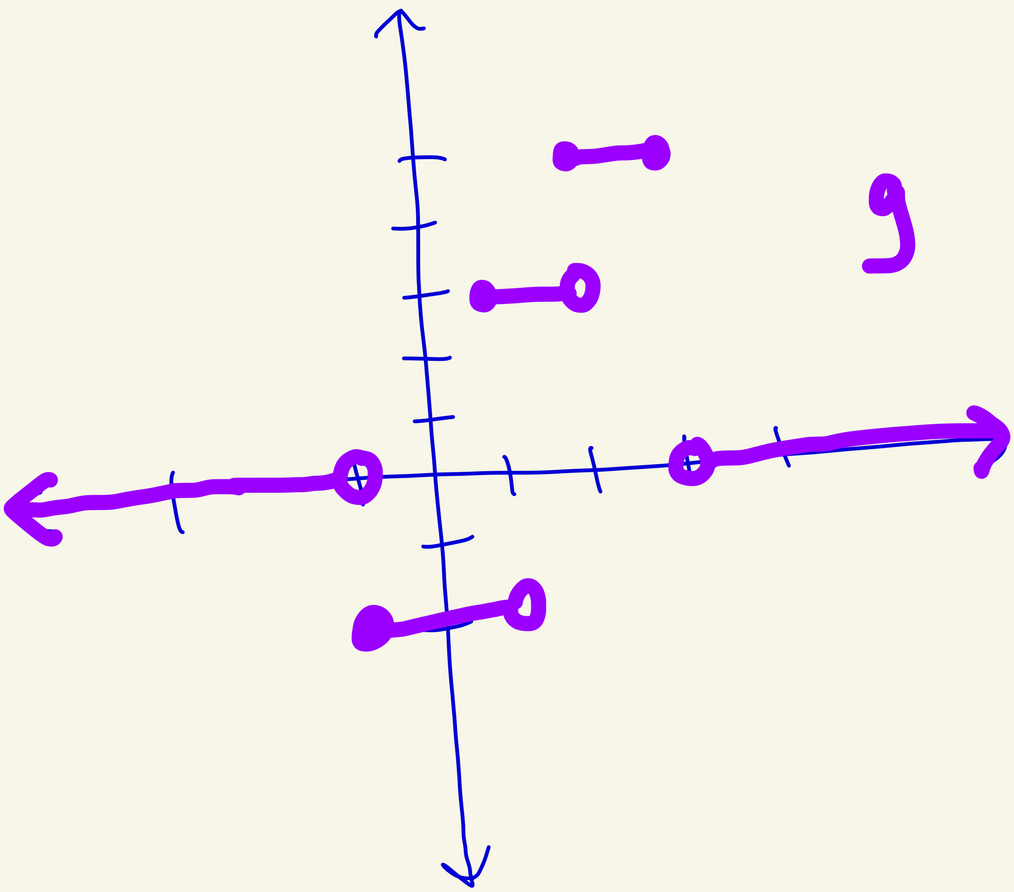
$$f = 2\cdot \chi_{[0,1)} + 5\cdot \chi_{[1,2]} + \chi_{(2,3)} - 2\cdot \chi_{[3,4)}$$

$$\int f = 2\cdot(1-0) + 5\cdot(2-1) + 1\cdot(3-2) - 2\cdot(4-3)$$

$$= 2 + 5 + 1 - 2 = 6$$

① (b) $g = -2 \cdot \chi_{[-1,2)} + 5 \cdot \chi_{[1,3]}$





$$g = -2 \cdot \chi_{[-1,1)} + 3 \cdot \chi_{[1,2)}$$

$$+ 5 \cdot \chi_{[2,3]}$$

$$\int g = -2 \cdot (1-(-1)) + 3 \cdot (2-1)$$

$$+ 5 \cdot (3-2)$$

$$= -2(2) + 3 + 5$$

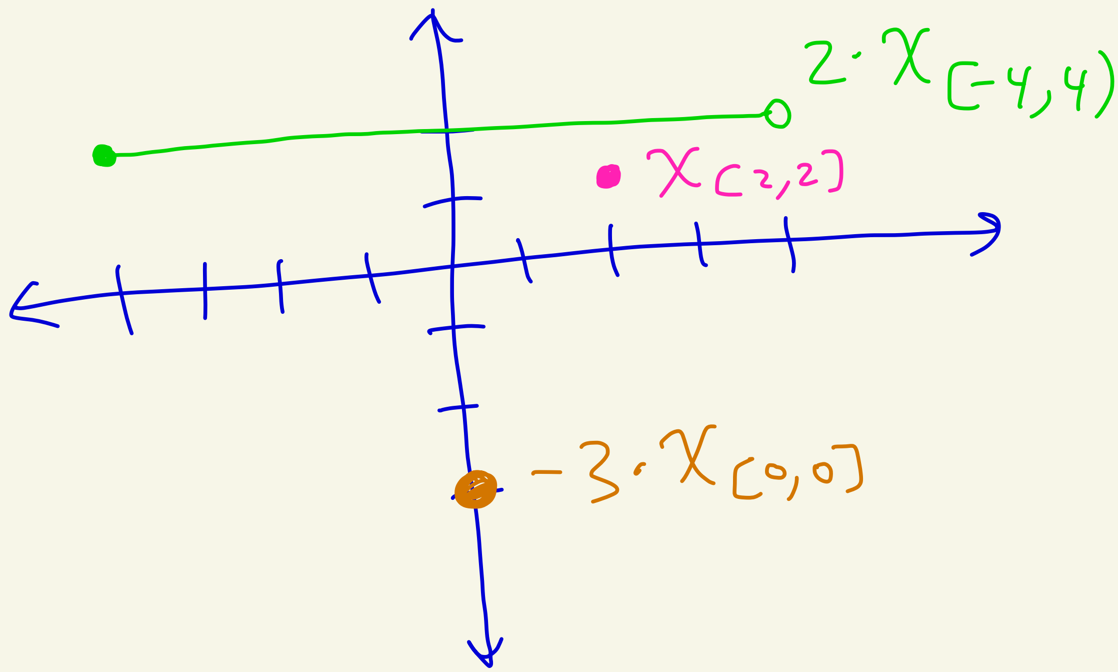
$$= 4$$

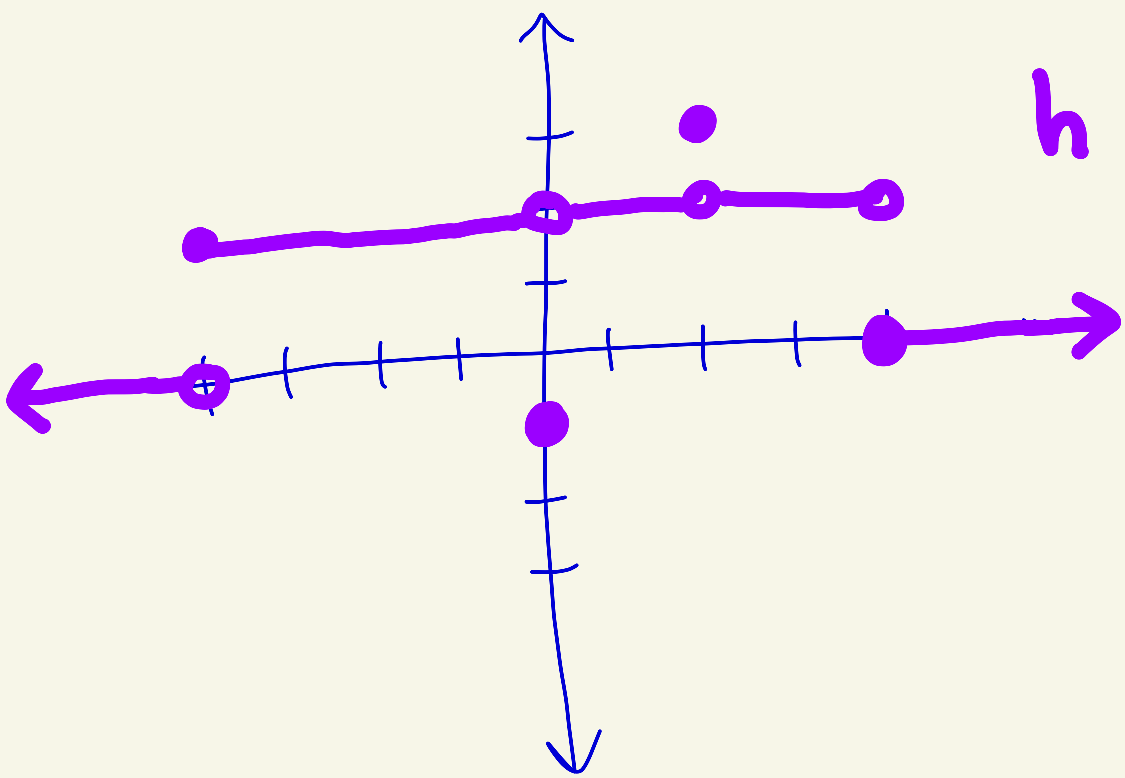
① (c)

$$h = 2\cdot\chi_{[-4,4)} + 4\pi\cdot\chi_{(-1,-1)} - 3\cdot\chi_{[0,0]} + \chi_{[2,2]}$$

Note that $\chi_{(-1,-1)} = \chi_{\phi}$ is the zero function

So, $h = 2\cdot\chi_{[-4,4)} - 3\cdot\chi_{[0,0]} + \chi_{[2,2]}$





$$h = 2 \cdot \chi_{[-4,0)} - \chi_{[0,0]} + 2 \cdot \chi_{(0,2)} + 3 \cdot \chi_{[2,2]} + 2 \cdot \chi_{(2,4)}$$

$$\int h = 2 \cdot (0-(-4)) - 1 \cdot (0-0) + 2 \cdot (2-0) + 3 \cdot (2-2) + 2 \cdot (4-2)$$

$$= 8 + 0 + 4 + 0 + 4$$

$$= 16$$

(2) Suppose $S \subseteq T$.

Let $x \in \mathbb{R}$.

<u>case 1:</u> Suppose $x \in S$.

So, $x \in T$ since $S \subseteq T$.

Then, $\chi_S(x) = 1$.

And $\chi_T(x) = 1$.

Thus, $\chi_S(x) \leq \chi_T(x)$.

<u>case 2:</u> Suppose $x \notin S$.

Then, $\chi_S(x) = 0$.

And $\chi_T(x) \geq 0$.

So, $\chi_S(x) \leq \chi_T(x)$.

In either case $\chi_S(x) \leq \chi_T(x)$.

$\blacksquare$

③ Let $x \in \mathbb{R}$.

Case 1: Suppose $x \in S$.

So, $\chi_S(x) = 1$.

Since $S = \bigcup_{k=1}^{n} S_k$ we know

$x \in S_i$ for some i with $1 \leq i \leq n$.

So, $\chi_{S_i}(x) = 1$.

Thus,

$$\chi_S(x) = 1 = \chi_{S_i}(x) \leq \sum_{k=1}^{n} \chi_{S_k}(x)$$

Case 2: Suppose $x \notin S$.

Then, $\chi_S(x) = 0$

Because $S = \bigcup_{k=1}^{n} S_k$ we know

$x \notin S_k$ for all k.

Thus, $\chi_{S_k}(x) = 0$ for $1 \le k \le n$.

Thus,

$$\chi_S(x) = 0 = \sum_{k=1}^{n} \chi_{S_k}(x)$$

In either case

$$\chi_S(x) \le \sum_{k=1}^{n} \chi_{S_k}(x).$$

$\blacksquare$

(4) Suppose that $A_1, A_2, \ldots, A_r$ are disjoint sets.

($\Rightarrow$) Suppose that $S = \bigcup_{i=1}^{r} A_i$

Let $x \in \mathbb{R}$.

Case 1: Suppose that $x \notin S$.

Then since $S = \bigcup_{i=1}^{r} A_i$. We know that $x \notin A_i$. Thus,

$$\chi_S(x) = 0 = 0 + 0 + \cdots + 0$$
$$= \chi_{A_1}(x) + \chi_{A_2}(x) + \cdots + \chi_{A_r}(x)$$

Case 2: Suppose $x \in S$.

Then since the A_i are disjoint and $S = \bigcup_{i=1}^{r} A_i$ we know that $x \in A_n$ for exactly one n with $1 \leq n \leq r$.

So,

$$\chi_S(x) = 1 = 0 + 0 + \dots + 1 + \dots + 0 + 0$$
$$= \chi_{A_1}(x) + \chi_{A_2}(x) + \dots + \chi_{A_n}(x) + \dots + \chi_{A_{r-1}}(x) + \chi_{A_r}(x)$$

Thus, from case 1 and case 2, $\chi_S = \sum_{i=1}^{r} \chi_{A_i}$

($\Leftarrow$) Suppose that

$$\chi_S = \chi_{A_1} + \chi_{A_2} + \dots + \chi_{A_r}$$

We will show that $S = \bigcup_{i=1}^{r} A_i$

Let $x \in S$.

Then $\chi_S(x) = 1$.

So, $\chi_{A_1}(x) + \chi_{A_2}(x) + \dots + \chi_{A_r}(x) = 1$.

Thus, $\chi_{A_n}(x) = 1$ for exactly one n where $1 \leq n \leq r$.

Thus $x \in A_n$.

So, $x \in \bigcup_{i=1}^{r} A_i$

Thus, $S \subseteq \bigcup_{i=1}^{r} A_i$.

Now suppose $y \in \bigcup_{i=1}^{r} A_i$

Then $y \in A_m$ for at least one m.

So, $\chi_{A_m}(y) = 1$.

Thus,

$$\chi_S(y) = \chi_{A_1}(y) + \dots + \chi_{A_m}(y) + \dots + \chi_{A_r}(y)$$
$$\geq 1.$$

So, $\chi_S(y) = 1$ since its either 0 or 1.

Thus, $y \in S$.

So, $\bigcup_{i=1}^{r} A_i \subseteq S$.

From above, $S = \bigcup_{i=1}^{r} A_i$ $\blacksquare$

(5) (a) Let f be a step function.

Then $f = a_1 \chi_{I_1} + a_2 \chi_{I_2} + \dots + a_r \chi_{I_r}$

where $I_1, I_2, \dots, I_r$ are disjoint bounded intervals and $a_1, \dots, a_r$ are non-zero real numbers.

We will show that $|f| = |a_1| \chi_{I_1} + \dots + |a_r| \chi_{I_r}$ and hence is a step function.

Let $x \in \mathbb{R}$.

<u>Case 1:</u> Suppose $f(x) = 0$.

Since $I_1, I_2, \dots, I_r$ are disjoint, either $x \in I_k$ for exactly one k, <u>or</u> $x \notin I_i$ for all i.

If $x \in I_k$ for exactly one k, then

$0 = f(x) = a_1 \chi_{I_1}(x) + \dots + a_r \chi_{I_r}(x) = a_k$

But none of the a_i are zero.

Thus, $x \notin I_i$ for all i and hence $\chi_{I_i}(x) = 0$ for all i.

Thus, for this x we have

$$
\begin{aligned}
|f|(x) &= |f(x)| \\
&= 0 \\
&= |a_1|\cdot 0 + |a_2|\cdot 0 + \dots + |a_r|\cdot 0 \\
&= |a_1|\chi_{I_1}(x) + \dots + |a_r|\cdot \chi_{I_r}(x).
\end{aligned}
$$

<u>Case 2</u>: Suppose $f(x) \neq 0$.

Since $f(x) = a_1\chi_{I_1}(x) + \dots + a_r\chi_{I_r}(x)$ we must have that $\chi_{I_k}(x) \neq 0$ for some k. Since $I_1, I_2, \dots, I_r$ are disjoint we must then have that $x \notin I_i$ if $i \neq k$.

Thus,

$$
\begin{aligned}
f(x) &= a_1\chi_{I_1}(x) + \dots + a_k\chi_{I_k}(x) + \dots + a_r\chi_{I_r}(x) \\
&= 0 + \dots + a_k\cdot 1 + \dots + 0 \\
&= a_k
\end{aligned}
$$

So,

$$|f|(x) = |f(x)|$$
$$= |a_k|$$
$$= |a_1|\cdot 0 + |a_2|\cdot 0 + \dots + |a_k|\cdot 1 + \dots + |a_{r-1}|\cdot 0 + |a_r|\cdot 0$$
$$= |a_1|\chi_{I_1}(x) + \dots + |a_k|\cdot \chi_{I_k}(x) + \dots + |a_r|\cdot \chi_{I_r}(x)$$

Thus, in either case $f = \sum_{j=1}^{r} |a_j| \chi_{I_j}$

So, f is a step function. $\blacksquare$

5 (b)

Let χ_1, χ_2 be step functions.

Then from class $\frac{1}{2}\chi_1 + \frac{1}{2}\chi_2$ and

$\chi_1 - \chi_2$ are step functions.

Thus, from 6(a), $\frac{1}{2}|\chi_1 - \chi_2|$

is a step function.

Hence, $\frac{1}{2}\chi_1 + \frac{1}{2}\chi_2 + \frac{1}{2}|\chi_1 - \chi_2|$

is a step function.

Let $f = \max\{\chi_1, \chi_2\}$.

We will show that

$$f = \frac{1}{2}\chi_1 + \frac{1}{2}\chi_2 + \frac{1}{2}|\chi_1 - \chi_2|$$

and thus f is a step

function which completes the proof.

Before we prove this, recall that

$$|a-b| = \begin{cases} a-b & \text{if } a-b \geq 0 \\ b-a & \text{if } a-b < 0 \end{cases}$$

$\boxed{\text{Claim}}$: $f = \frac{1}{2}\chi_1 + \frac{1}{2}\chi_2 + \frac{1}{2}|\chi_1 - \chi_2|$

Let $x \in \mathbb{R}$.

<u>case 1:</u> Suppose $\chi_1(x) \geq \chi_2(x)$.

Then $f(x) = \left(\max\{\chi_1, \chi_2\}\right)(x) = \chi_1(x)$.

And, $|\chi_1 - \chi_2|(x) = \chi_1(x) - \chi_2(x)$.

$\chi_1(x) \geq \chi_2(x)$
$\chi_1(x) - \chi_2(x) \geq 0$

So,

$$\frac{1}{2}\chi_1(x) + \frac{1}{2}\chi_2(x) + \frac{1}{2}|\chi_1 - \chi_2|(x)$$

$$= \frac{1}{2}\chi_1(x) + \frac{1}{2}\chi_2(x) + \frac{1}{2}\chi_1(x) - \frac{1}{2}\chi_2(x)$$

$$= \chi_1(x) = f(x).$$

Case 2: Suppose $x_2(x) > x_1(x)$.

Then $f(x) = (\max\{x_1, x_2\})(x) = x_2(x)$.

And, $|x_1 - x_2|(x) = x_2(x) - x_1(x)$.

$\boxed{\begin{array}{l} x_2(x) > x_1(x) \\ x_1(x) - x_2(x) < 0 \end{array}}$

So,

$$\frac{1}{2}x_1(x) + \frac{1}{2}x_2(x) + \frac{1}{2}|x_1 - x_2|(x)$$
$$= \frac{1}{2}x_1(x) + \frac{1}{2}x_2(x) + \frac{1}{2}x_2(x) - \frac{1}{2}x_1(x)$$
$$= x_2(x) = f(x).$$

Thus, in either case

$$f = \frac{1}{2}x_1 + \frac{1}{2}x_2 + \frac{1}{2}|x_1 - x_2|$$

$\blacksquare$

5 (c)

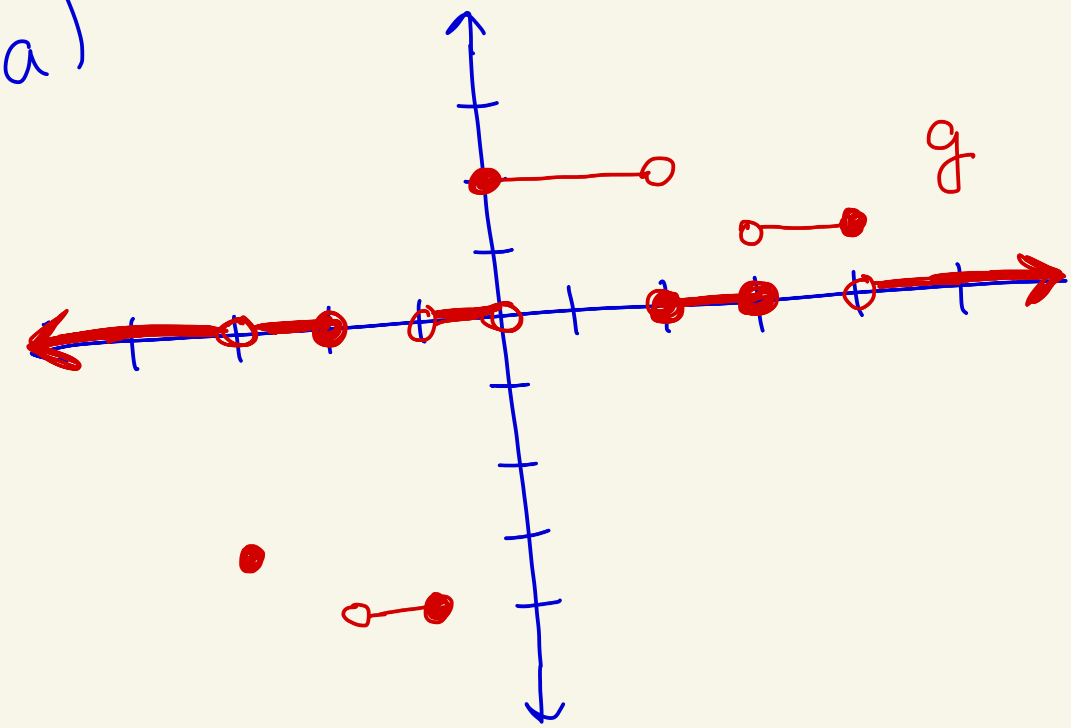
This proof is similar to 6(b) except use the fact that

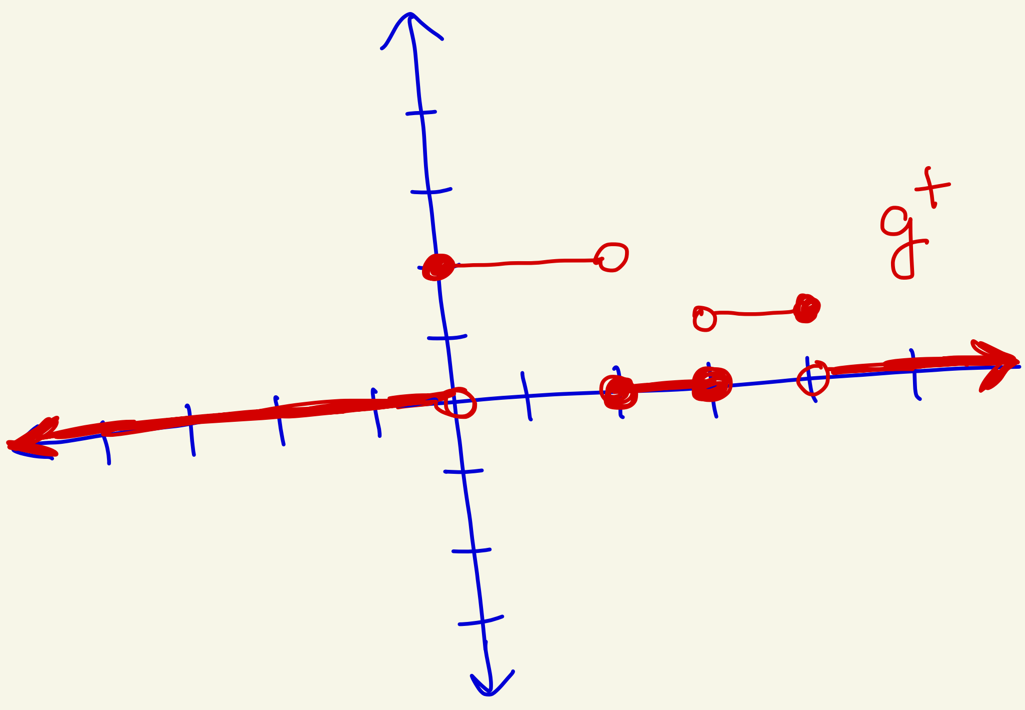
$$\min\{x_1, x_2\} = \frac{1}{2}f + \frac{1}{2}g - \frac{1}{2}|f-g|$$

Try it and use the proof of 6(b) as a guide.

$\blacksquare$

6 (a)





$$g^+ = 2 \cdot \chi_{[0,2)} + \chi_{(3,4]}$$

⑥(b) $\boxed{\text{Method 1}}$

Let f be a step function.
By grouping the positive and negative terms together we may write

$$f = b_1 \chi_{I_1} + \cdots + b_t \chi_{I_t} + c_1 \chi_{J_1} + \cdots + c_s \chi_{J_s}$$

where $b_1, \ldots, b_t, c_1, \ldots, c_s \in \mathbb{R}$ with $b_1, b_2, \ldots, b_t > 0$ and $c_1, c_2, \ldots, c_s < 0$ and $I_1, I_2, \ldots, I_t, J_1, J_2, \ldots, J_s$ are bounded disjoint intervals.

Then f is positive on $I_1, I_2, \ldots, I_t$ and f is negative on $J_1, J_2, \ldots, J_s$.

Thus,

$$f^+ = b_1 \chi_{I_1} + \cdots + b_t \chi_{I_t}$$

So, f^+ is a step function. $\boxed{\text{method 1}}$

Method 2 for 6(b)

Claim: $f^+ = \frac{1}{2}|f| + \frac{1}{2}f$

pf of claim: Let $x \in \mathbb{R}$.

Suppose $f(x) > 0$.
Then, $f^+(x) = f(x)$
And $|f|(x) = |f(x)| = f(x)$.
So, $f^+(x) = f(x) = \frac{1}{2}|f(x)| + \frac{1}{2}f(x) = \frac{1}{2}|f|(x) + \frac{1}{2}f(x)$.

Suppose $f(x) \leq 0$.
Then, $f^+(x) = 0$.
And $|f|(x) = |f(x)| = -f(x)$.
Thus, $f^+(x) = 0 = -\frac{1}{2}f(x) + \frac{1}{2}f(x)$
$= \frac{1}{2}|f(x)| + \frac{1}{2}f(x) = \frac{1}{2}|f|(x) + \frac{1}{2}f(x)$. ∎

By problem 5, $|f|$ is a step function.
Thus, $\frac{1}{2}|f| + \frac{1}{2}f$ is a step function.
Thus, f^+ is a step function

$\boxed{\text{Method 2}}$

7 (a)

($\Rightarrow$) Let $S \in \mathcal{R}$.

Then $S = I_1 \cup I_2 \cup \cdots \cup I_r$ where $I_1, I_2, \ldots, I_r$ are disjoint bounded intervals.

By problem 5, we have that

$$\chi_S = \chi_{I_1} + \chi_{I_2} + \cdots + \chi_{I_r}$$

Since the I_k are bounded intervals, this shows that χ_S is a step function.

($\Leftarrow$) Suppose χ_S is a step function. Then from class, we can write

$$\chi_S = a_1 \chi_{I_1} + a_2 \chi_{I_2} + \cdots + a_r \chi_{I_r}$$

where $a_1, \ldots, a_r$ are non-zero real numbers and $I_1, I_2, \ldots, I_r$ are disjoint bounded intervals.

Pick some n with $1 \le n \le r$.

Suppose $x \in I_n$.

Then since the sets $I_1, I_2, \ldots, I_r$ are disjoint we know that $\chi_n(x) = 1$ but $\chi_k(x) = 0$ for $k \neq n$.

Since $\chi_S = a_1 \chi_{I_1} + a_2 \chi_{I_2} + \ldots + a_r \chi_{I_r}$

this means that

$$\chi_S(x) = a_1 \cdot 0 + \ldots + a_n \cdot 1 + \ldots + a_r \cdot 0$$
$$= a_n$$

Since $a_n \neq 0$ and $\chi_S(x) = 0$ or 1 we know that $\chi_S(x) = 1$ and hence $a_n = 1$.

Since n was arbitrary this shows that

$$\chi_S = \chi_{I_1} + \chi_{I_2} + \ldots + \chi_{I_r}$$

By problem 5, $S = \bigcup_{i=1}^{r} I_r$

and hence $S \in \mathcal{R}$ $\boxtimes$

7 (b)

Let $S, T \in \mathcal{R}$.

Then by 7(a), χ_S and χ_T are step functions.

Thus, by problems 5 and 6 we have that $\max\{\chi_S, \chi_T\}$, $\min\{\chi_S, \chi_T\}$ and $(\chi_S - \chi_T)^+$ are step functions.

We will show that

$$\chi_{S \cup T} = \max\{\chi_S, \chi_T\}$$

$$\chi_{S \cap T} = \min\{\chi_S, \chi_T\}$$

$$\chi_{S-T} = (\chi_S - \chi_T)^+$$

and thus by 7(a) we have that $S \cap T, S \cap T, S - T \in \mathcal{R}$

Claim 1: $\chi_{S\cup T} = \max\{\chi_S, \chi_T\}$

Let $x \in S \cup T$. Then $\chi_{S\cup T}(x) = 1$.

Since $x \in S$ or $x \in T$ [or both],

either $\chi_S(x) = 1$ or $\chi_T(x) = 1$ [or both].

So, $\chi_{S\cup T}(x) = 1 = \max\{\chi_S(x), \chi_T(x)\}$

$(\max\{\chi_S, \chi_T\})(x)$

Now suppose $x \notin S \cup T$. So, $x \notin S$ and $x \notin T$.

Then $\chi_{S\cup T}(x) = 0$.

And $\chi_S(x) = 0$ and $\chi_T(x) = 0$.

So, $\chi_{S\cup T}(x) = 0 = \max\{0, 0\}$

$= \max\{\chi_S(x), \chi_T(x)\}$

$(\max\{\chi_S, \chi_T\})(x)$

Thus, $\chi_{S\cup T} = \max\{\chi_S, \chi_T\}$

Claim 2: $\chi_{S\cap T} = \min\{\chi_S, \chi_T\}$

Suppose $x \in S \cap T$.

Then, $x \in S$ and $x \in T$.

So, $\chi_{S\cap T}(x) = \chi_S(x) = \chi_T(x) = 1$.

Thus, $\chi_{S\cap T}(x) = 1 = \min\{1, 1\}$

$$= \min\{\chi_S(x), \chi_T(x)\}$$

$$= (\min\{\chi_S, \chi_T\})(x)$$

Suppose now that $x \notin S \cap T$.

Then either

(i) $x \notin S \cap T$, $x \in S$, $x \notin T$

(ii) $x \notin S \cap T$, $x \notin S$, $x \in T$

or (iii) $x \notin S \cap T$, $x \notin S$, $x \notin T$.

If (i) is true, then

$\chi_{S\cap T}(x) = 0 = \min\{1, 0\}$

$= \min\{\chi_S(x), \chi_T(x)\} = (\min\{\chi_S, \chi_T\})(x)$

If (ii) is true, then

$$\chi_{S\cap T}(x) = 0 = \min\{0, 1\}$$
$$= \min\{\chi_S(x), \chi_T(x)\} = (\min\{\chi_S, \chi_T\})(x)$$

If (iii) is true, then

$$\chi_{S\cap T}(x) = 0 = \min\{0, 0\}$$
$$= \min\{\chi_S(x), \chi_T(x)\} = (\min\{\chi_S, \chi_T\})(x)$$

Hence, from all the above cases we have that $\chi_{S\cap T} = \min\{\chi_S, \chi_T\}$.

Claim 3: $\chi_{S-T} = (\chi_S - \chi_T)^+$

Suppose $x \in S-T$.

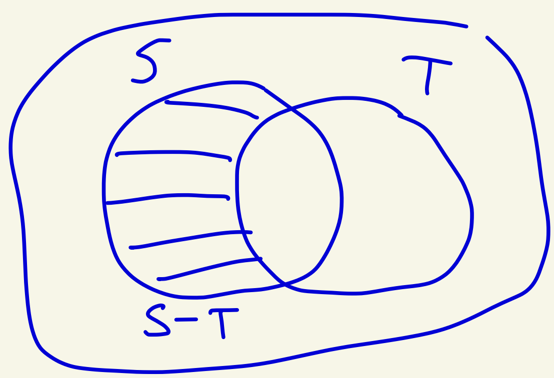
Then $x \in S$ and $x \notin T$.

So, $\chi_{S-T}(x) = 1$.

And, $\chi_S(x) - \chi_T(x) = 1 - 0 = 1$.

So, $(\chi_S - \chi_T)^+(x) = 1$.

Thus, $\chi_{S-T}(x) = (\chi_S - \chi_T)^+(x)$.



Now suppose $x \notin S-T$.

Then either

(i) $x \in S$ and $x \in T$

(ii) $x \notin S$ and $x \in T$

or (iii) $x \notin S$ and $x \notin T$

If (i) is true, then $\chi_{S-T}(x) = 0$ and $\chi_S(x) - \chi_T(x) = 1 - 1 = 0$. So,

$\chi_{S-T}(x) = 0 = (\chi_S - \chi_T)^+(x)$.

If (ii) is true, then $\chi_{S-T}(x) = 0$ and

$\chi_S(x) - \chi_T(x) = 0 - 1 = -1$. So,

$\chi_{S-T}(x) = 0 = (\chi_S - \chi_T)^+(x)$.

If (iii) is true, then $\chi_{S-T}(x) = 0$ and

$\chi_S(x) - \chi_T(x) = 0 - 0 = 0$. So,

$\chi_{S-T}(x) = 0 = (\chi_S - \chi_T)^+(x)$.

In all the above cases

$$\chi_{S-T} = (\chi_S - \chi_T)^+$$

$\blacksquare$

(7) (c) Let $S, T \in \mathcal{R}$.
And suppose $S \subseteq T$.

From class,
$$\ell(S) = \int \chi_S \quad \text{and} \quad \ell(T) = \int \chi_T$$

From problem 2, we know that since $S \subseteq T$ we have that
$$\int \chi_S \leq \int \chi_T.$$

Thus, $\ell(S) \leq \ell(T)$. $\blacksquare$

7 (d)

Let $A_1, A_2, \ldots, A_s$ be bounded intervals.

Then $A_1 \in \mathcal{R}$, $A_2 \in \mathcal{R}$, $\ldots$, $A_s \in \mathcal{R}$.

Since by part (b), $\mathcal{R}$ is closed under union we have that

$$A = \bigcup_{i=1}^{s} A_i \in \mathcal{R}.$$

7 (e)

Let $I_1, I_2, \ldots, I_r$ be disjoint bounded intervals.

Suppose that there exist bounded intervals $J_1, J_2, \ldots, J_t$ where

$$\bigcup_{j=1}^{r} I_j \subseteq \bigcup_{i=1}^{t} J_i$$

Let $S = \bigcup_{j=1}^{r} I_j$

Then by 7(d), $S \in \mathcal{R}$.

And thus $\int \chi_S = \sum_{j=1}^{r} l(I_j)$.

We are given that $S \subseteq \bigcup_{i=1}^{t} J_i$.

Claim: $\chi_S(x) \leq \sum_{i=1}^{t} \chi_{J_i}$

proof of claim: Let $x \in \mathbb{R}$.

Suppose $x \in S$. Then $x \in \bigcup_{i=1}^{t} J_i$ since $S \subseteq \bigcup_{i=1}^{t} J_i$

So $x \in J_m$ for at least one m with $1 \leq m \leq t$.

So, $\chi_{J_m}(x) = 1$.

Thus, $\underbrace{\chi_S(x) = 1}_{x \in S} = \chi_{J_m}(x) \leq \sum_{i=1}^{t} \chi_{J_i}(x)$.

Now suppose $x \notin S$.

Then, $\chi_S(x) = 0$.

So, $\chi_S(x) = 0 \leq \sum_{i=1}^{t} \chi_{J_i}(x)$

$\chi_{J_i}(x) \geq 0$ for each i

$\boxed{\text{claim}}$

Now integrating the equation in the claim gives that

$$\sum_{j=1}^{r} \ell(I_j) = \int \chi_S \leq \int \sum_{i=1}^{t} \chi_{J_i}$$

$$= \sum_{i=1}^{t} \ell(J_i).$$

$\boxed{\not\!/\!/}$