Homework #9

① Check if the following are linear transformations. If so, prove it. If not give an explicit example why not.

(a) $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(x,y) = (x+2y, 3x-y)$

(b) $T: \mathbb{R}^3 \to \mathbb{R}^2$ given by $T(x,y,z) = (2x-y+z, y-4z)$

(c) $T: \mathbb{R}^2 \to \mathbb{R}$ given by $\underbrace{T(x,y) = \sqrt{x^2+y^2}}_{\text{This is the same as } T(\vec{v}) = \|\vec{v}\|}$

(d) $T: M_{2,2} \to \mathbb{R}$ given by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = 3a - 4b + c - d$$

(e) $T: M_{2,2} \to \mathbb{R}$ given by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a^2 + b^2$$

(f) $T: P_2 \to P_2$ given by

$$T(a_0 + a_1 x + a_2 x^2) = a_0 + a_1(1+x) + a_2(1+x)^2$$

(g) $T: P_2 \to P_2$ given by

$$T(a_0 + a_1 x + a_2 x^2) = (1+a_0) + (1+a_1)x + (1+a_2)x^2$$

② Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Suppose we know that $T(1,0,0) = (2,1,-1)$, $T(0,1,0) = (0, \pi, 2/3)$, and $T(0,0,1) = (-1,0,0)$. Find a formula for $T(x,y,z)$ in general.

③ Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ -2x+4y \end{pmatrix}$. T is a linear transformation (you can verify that for extra practice).

Let $\beta = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right]$. Then β is an ordered basis for $\mathbb{R}^2$ (you can verify for extra practice).

(a) Find $[T]_\beta$

(b) Verify that $[T]_\beta [\vec{x}]_\beta = [T(\vec{x})]_\beta$
for $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{x} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

④ Let $T: P_2 \to P_1$ be the linear transformation defined by

$$T(a_0 + a_1 x + a_2 x^2) = (a_0 + a_1) - (2a_1 - 3a_2)x$$

Let $\beta = [1, x, x^2]$ and $\beta' = [1, x]$ be ordered bases for P_2 and P_1 respectively.

(a) Calculate $[T]_\beta^{\beta'}$

(b) Show that $[T(\vec{v})]_{\beta'} = [T]_\beta^{\beta'} [\vec{v}]_\beta$

for $\vec{v} = 1 + x + x^2$ and $\vec{v} = 2x^2$.

(5) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - y \\ x + y \end{pmatrix}$$

and let $\beta = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right]$ be an ordered basis for $\mathbb{R}^2$.

(a) Calculate $[T]_\beta$

(b) Verify that $[T(\vec{v})]_\beta = [T]_\beta [\vec{v}]_\beta$ for $\vec{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(6) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator defined by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x - 2y \\ -y \end{pmatrix}$$

Let $\beta = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]$ and $\beta' = \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \end{pmatrix} \right]$ be ordered bases for $\mathbb{R}^2$.

(a) Calculate $[T]_\beta^{\beta'}$

(b) Calculate the change of basis matrix $Q = [I]_\beta^{\beta'}$ from β to β'.

(c) Show that $Q[\vec{v}]_\beta = [\vec{v}]_{\beta'}$ for $\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$.

(d) Show that $[T]_\beta = Q^{-1} [T]_{\beta'} Q$

(7) Calculate the change of basis matrix from β to β' where

(a) $V = \mathbb{R}^3$, $\beta = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right]$

and $\beta' = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right]$

(b) $V = \mathbb{R}^2$, $\beta = \left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right]$

$\beta' = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \end{pmatrix} \right]$.

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(8) (a) Using your answer $Q = [I]_\beta^{\beta'}$ from 7(a)
Show that $Q[\vec{v}]_\beta = [\vec{v}]_{\beta'}$ where
$\vec{v} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$.

(b) Using your answer $Q = [I]_\beta^{\beta'}$ from 7(b)
Show that $Q[\vec{v}]_\beta = [\vec{v}]_{\beta'}$ where
$\vec{v} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$.