Math 5800
Homework # 6
Sequences of functions and the standard construction

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1. Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ for $n \geq 1$ where

$$f_n(x) = x + \frac{(-1)^n}{n}$$

Let $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = x$$

for all $x \in \mathbb{R}$.

   (a) Draw a picture of f_1, f_2, f_3, f_4, and f on the same graph.
   (b) Write out the first 4 elements of the sequence $(f_n(1))_{n=1}^{\infty}$ and draw them on the previous graph from (a).
   (c) Prove that $(f_n(1))_{n=1}^{\infty}$ converges to 1.
   (d) Prove that the sequence f_n converges to f pointwise on all of $\mathbb{R}$.

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2. Consider the sequence of functions $f_n : \mathbb{R} \to \mathbb{R}$ for $n \geq 1$ where

$$f_n(x) = \begin{cases} x^2/n & \text{if } x \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}$$

Let

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \text{ or } x = -1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Draw a picture of f. Also draw a picture of f_1, f_2, and f_3 on the same graph.
   (b) Prove that the sequence f_n converges to f pointwise on all of $\mathbb{R} - \{1, -1\}$
   (c) Prove that f_n does not converge to f pointwise on $\{1, -1\}$.

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3. (Standard construction problem) Let

$$f(x) = \begin{cases} x+1 & \text{if } x \in [-1,1] \\ 0 & \text{otherwise} \end{cases}$$

   (a) Calculate explicitly the standard construction γ_1 and γ_2 for f on $[-1,1]$. Draw pictures of each.

   (b) Prove that γ_n converges pointwise to f on $[-1,1]$. That is, show that if $x \in [-1,1]$, then $\lim_{n\to\infty} \gamma_n(x) = f(x)$.

   [Hint: Use the fact that f is an increasing function to show that if $x \in [-1,1]$, then $|\gamma_n(x) - f(x)| \leq 1/2^{n-1}$.

   (c) Prove that γ_n converges pointwise to f on all of $\mathbb{R}$.

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4. (Standard construction problem) Let

$$f(x) = \begin{cases} x^2 & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

   (a) Calculate explicitly the standard construction γ_1 and γ_2 for f on $[0,1]$. Draw pictures of each.

   (b) Prove that γ_n converges pointwise to f on $[0,1]$. That is, show that if $x \in [0,1]$, then $\lim_{n\to\infty} \gamma_n(x) = f(x)$.

   [Hint: For $x \in [(k-1)/2^n, k/2^n)$ show that $|f(x) - \gamma_n(x)| < k^2/2^{2n} - (k-1)^2/2^{2n} < 1/2^{n-1}$.]

   (c) Prove that γ_n converges pointwise to f on all of $\mathbb{R}$.

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5. Suppose that $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ are sequences of functions with $f_n : \mathbb{R} \to \mathbb{R}$ and $g_n : \mathbb{R} \to \mathbb{R}$. Let $A \subseteq \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. If f_n converges to f pointwise on A and g_n converges to g pointwise on A, then $f_n + g_n$ converges to $f + g$ pointwise on A.

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6. Suppose that $(f_n)_{n=1}^{\infty}$ and $(g_n)_{n=1}^{\infty}$ are sequences of functions with $f_n : \mathbb{R} \to \mathbb{R}$ and $g_n : \mathbb{R} \to \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. If f_n converges to f almost everywhere on $\mathbb{R}$ and g_n converges to g almost everywhere on $\mathbb{R}$, then $f_n + g_n$ converges to $f + g$ almost everywhere on $\mathbb{R}$.
