Homework #5 Solutions

① There are 18 black numbers The probability of getting black one one spin is $p = \frac{18}{38} = \frac{9}{19}$.

(a) $p(0) = P(X=0) = \binom{5}{0} p^0 (1-p)^5 = 5\cdot\left(\frac{9}{19}\right)^0 \cdot \left(1-\frac{9}{19}\right)^5 =$

$= \frac{100{,}000}{2{,}476{,}099} \approx 0.04\ldots$

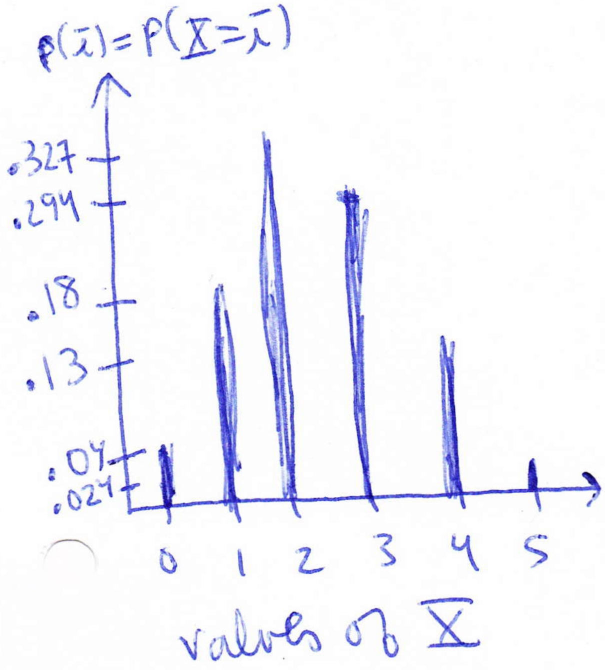
$p(1) = P(X=1) = \binom{5}{1}\left(\frac{9}{19}\right)^1\left(1-\frac{9}{19}\right)^{5-1} = \frac{450{,}000}{2{,}476{,}099} \approx 0.18\ldots$

$p(2) = P(X=2) = \binom{5}{2}\left(\frac{9}{19}\right)^2\left(1-\frac{9}{19}\right)^{5-2} = \frac{810{,}000}{2{,}476{,}099} \approx 0.327\ldots$

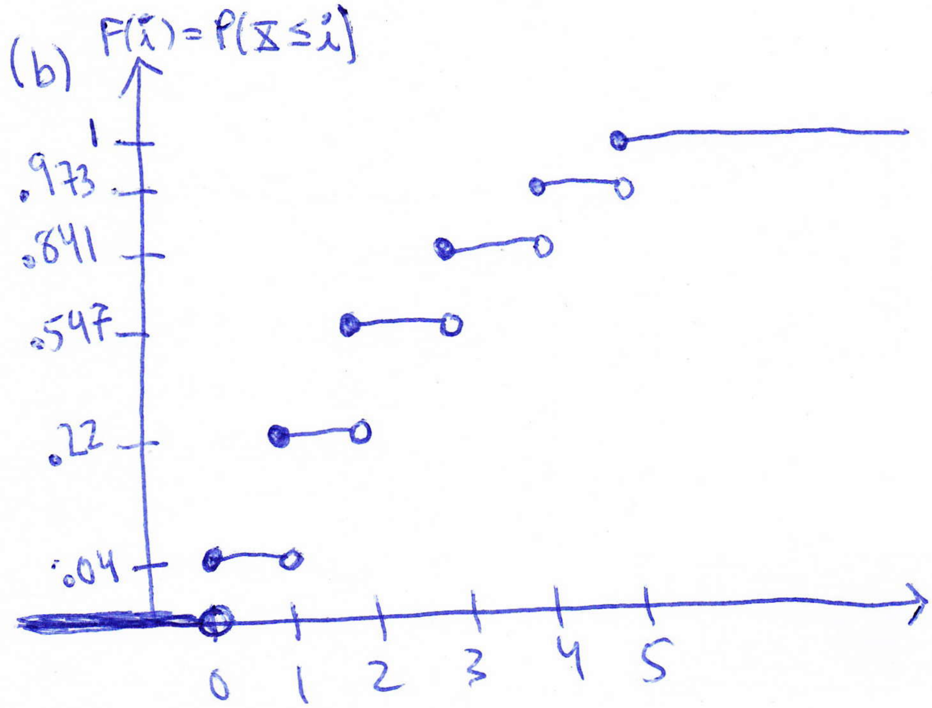
$p(3) = P(X=3) = \binom{5}{3}\left(\frac{9}{19}\right)^3\left(1-\frac{9}{19}\right)^{5-3} = \frac{729{,}000}{2{,}476{,}099} \approx 0.294\ldots$

$p(4) = P(X=4) = \binom{5}{4}\left(\frac{9}{19}\right)^4\left(1-\frac{9}{19}\right)^{5-4} = \frac{328{,}050}{2{,}476{,}099} \approx 0.132\ldots$

$p(5) = P(X=5) = \binom{5}{5}\left(\frac{9}{19}\right)^5\left(1-\frac{9}{19}\right)^{5-5} = \frac{59{,}049}{2{,}476{,}099} \approx 0.0238\ldots$



(b)



(c) $P(X \geq 3) = P(X=3) + P(X=4) + P(X=5)$

$$= \frac{1,116,099}{2,476,099} \approx 0.450799\ldots$$

(d) $E[X] = (0)\left(\frac{100,000}{2,476,099}\right) + (1)\left(\frac{450,000}{2,476,099}\right) +$

$$+ (2)\left(\frac{810,000}{2,476,099}\right) + (3)\left(\frac{729,000}{2,476,099}\right) + (4)\left(\frac{328,050}{2,476,099}\right) + (5)\left(\frac{59,049}{2,476,099}\right)$$

$$= \frac{45}{19} \approx 2.36842$$

(e) $Var[X] = E[X^2] - \left(E[X]\right)^2$

$$E[X^2] = (0)^2\left(\frac{100,000}{2,476,099}\right) + (1)^2\left(\frac{450,000}{2,476,099}\right) + (2)^2\left(\frac{810,000}{2,476,099}\right) +$$

$$(3)^2\left(\frac{729,000}{2,476,099}\right) + (4)^2\left(\frac{328,050}{2,476,099}\right) + (5)^2\left(\frac{59,049}{2,476,099}\right)$$

$$= \frac{2475}{361}$$

$$Var[X] = \frac{2475}{361} - \left(\frac{45}{19}\right)^2 = \frac{450}{361}$$

Note: We could have gotten (d) and (e) by

$E[X] = np$ and $Var[X] = np(1-p)$

where $n=5$, $p = \frac{9}{19}$. (These are the formulas for binomial random variables)

2. The probability of rolling a seven or an eleven $\boxed{\text{on one roll is}}$ $p = \frac{6}{36} + \frac{2}{36} = \frac{8}{36} = \frac{2}{9} \approx 0.2222\ldots$

(a) $E[X] = n \cdot p = (10)\left(\frac{2}{9}\right) = \frac{20}{9} \approx 2.22\ldots$ ← formulas for binomial random variables

$Var[X] = np(1-p) = (10)\left(\frac{2}{9}\right)\left(1-\frac{2}{9}\right) = \frac{140}{81} \approx 1.7284$

(b) $P(X=5) = \binom{10}{5}\left(\frac{2}{9}\right)^5\left(1-\frac{2}{9}\right)^{10-5} = \frac{15,059,072}{387,420,489}$

$\approx 0.0388701\ldots$

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3. The probability that a double six appears on one throw of the die is $p = \frac{1}{36}$.

We want

$P(X \geq 3) = P(X=3) + P(X=4) + P(X=5) + \ldots + P(X=10)$

Lets instead calculate $1 - P(X \geq 3)$ which is

$P(X=0) + P(X=1) + P(X=2) = \binom{10}{0}\left(\frac{1}{36}\right)^0\left(1-\frac{1}{36}\right)^{10} + \binom{10}{1}\left(\frac{1}{36}\right)^1\left(1-\frac{1}{36}\right)^9 + \binom{10}{2}\left(\frac{1}{36}\right)^2\left(1-\frac{1}{36}\right)^8$

$\frac{11,259,376,953,125}{11,284,439,629,824} \approx 0.997779$

Answer is $1 - \#$
$\approx 0.00222099\ldots$

(4) On each flip of the coin we have probability $p = \frac{1}{2}$ that a heads will occur.

(a) $E[X] = np = (15)(\frac{1}{2}) = \frac{15}{2} = 7.5$

(b) $\binom{15}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{15-3} = \frac{15!}{3!\,12!} \cdot \frac{1}{8} \cdot \frac{1}{4096}$

$= \frac{15 \cdot 14 \cdot 13}{3 \cdot 2 \cdot 1} \cdot \frac{1}{8} \cdot \frac{1}{4096}$

$= \frac{455}{32{,}768} \approx 0.0138855\ldots$

(c) $P(X=0) + P(X=1) + P(X=2)$

$= \binom{15}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{15-0} + \binom{15}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{15-1} + \binom{15}{2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{15-2}$

$= 1 \cdot 1 \cdot \frac{1}{32{,}768} + 15 \cdot \frac{1}{2} \cdot \frac{1}{16{,}384} + 105 \cdot \frac{1}{4} \cdot \frac{1}{8192}$

$= \frac{121}{32{,}768} \approx 0.00369263\ldots$

(d) We want $P(X \geq 2)$. This would involve calculating $P(X \geq 2) = P(X=2) + P(X=3) + \ldots + P(X=15)$.
Instead we do

$P(X \geq 2) = 1 - P(X<2) = 1 - P(X=0) - P(X=1)$

$= 1 - \binom{15}{0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{15} - \binom{15}{1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{14} = 1 - \frac{1}{2^{15}} - 15 \cdot \frac{1}{2^{15}}$

$= \frac{32768 - 1 - 15}{2^{15}} = \frac{32752}{32768} \approx 0.999512\ldots$