Math 5680

Homework # 4 - Part 1

Laurent series and residues

1. For each part, do the following: (i) Draw a picture of A, (ii) find the Laurent series expansion for $f(z)$ around z_0 in the indicated region A, (iii) what kind of singularity is it at z_0 ?, and (iv) what is the residue at z_0 ?

   (a) $f(z) = \dfrac{1}{z(z+1)}$, $z_0 = 0$, $A = \{z \mid 0 < |z| < 1\}$

   (b) $f(z) = \dfrac{z}{(z+1)}$, $z_0 = 0$, $A = \{z \mid 0 < |z| < 1\}$

   (c) $f(z) = \dfrac{3\, e^z}{z^2}$, $z_0 = 0$, $A = \{z \mid 0 < |z|\} = \mathbb{C} - \{0\}$

2. Answer the same questions as in problem 1, but in this one you figure out what A will be.

   (a) $f(z) = \dfrac{z}{(z+1)}$, $z_0 = -1$

3. Find the Laurent series for $f(z) = \dfrac{1}{z(z+1)}$ inside of $A = \{z \mid 1 < |z|\}$.

4. (a) Expand $f(z) = \dfrac{1}{z(z-1)(z-2)}$ into a Laurent series in the region $A = \{z \mid 0 < |z| < 1\}$.

   (b) Expand the same function in the region $B = \{z \mid 1 < |z| < 2\}$.

5. Expand $f(z) = \dfrac{1}{z^2(1-z)}$, into a Laurent series in the region $A = \{z \mid 0 < |z-1| < 1\}$. Classify the pole and give the residue at $z_0 = 1$.

6. Find the Laurent series for $f(z) = \dfrac{z+1}{z^3(z^2+1)}$ on the region $A = \{z \mid 0 < |z| < 1\}$. Calculate the residue at $z_0 = 0$.

7. Consider the Laurent series of $f(z) = \dfrac{e^{1/z}}{(1-z)}$ in the region $A = \{z \mid 0 < |z| < 1\}$. Compute the terms b_2, b_1, a_0, and a_1 of the series.

8. Classify the pole of the function at z_0.

   (a) $f(z) = \dfrac{1}{(1-z)^2}$, $z_0 = 1$

   (b) $f(z) = \dfrac{\sin(z-1)}{z^2}$, $z_0 = 0$

9. Consider the function $f(z) = 1/(e^z - 1)$. Show that f has a simple pole at $z_0 = 0$. Find the first few terms of the Laurent series b_1, a_0, and a_1.

10. Find the residue of f at z_0.

    (a) $f(z) = \dfrac{1}{z^2 - 1}$, $z_0 = 1$

    (b) $f(z) = \dfrac{z}{z^2 - 1}$, $z_0 = 1$

    (c) $f(z) = \dfrac{e^z - 1}{z^2}$, $z_0 = 0$

    (d) $f(z) = \dfrac{e^z - 1}{z}$, $z_0 = 0$

11. Suppose that f is analytic at z_0 and has a zero of multiplicity k at z_0. Show that the residue of $g(z) = \dfrac{f'(z)}{f(z)}$ at z_0 is k.