Math 4740

HW 4 Solutions

①

(a) Size of sample space $|S| = 6^3$

You lose $-\$1$ if none of the dice match your number. Thus,

$$\boxed{p(-1) = P(X=-1) = \frac{5\cdot 5\cdot 5}{6^3} = \frac{125}{216}}$$

You win $\$1$ if exactly one die matches your number. Thus,

$$\boxed{p(1) = P(X=1) = \frac{1\cdot 5\cdot 5 + 5\cdot 1\cdot 5 + 5\cdot 5\cdot 1}{6^3} = \frac{75}{216}}$$

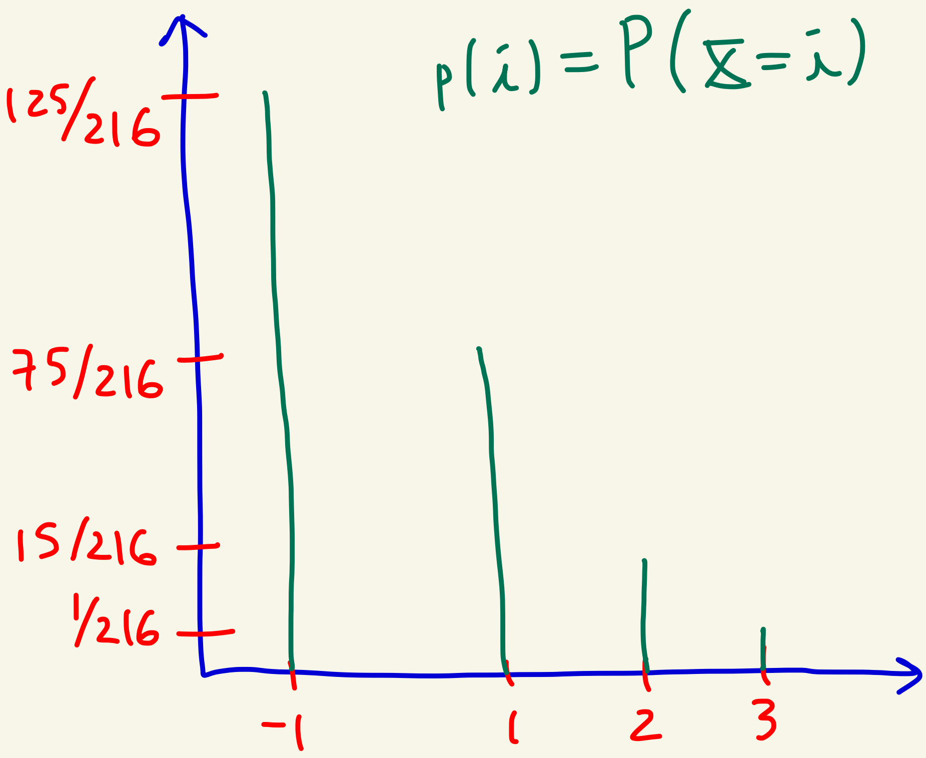
You win $\$2$ if exactly two dice match your number. Thus,

$$\boxed{p(2) = P(X=2) = \frac{1\cdot 1\cdot 5 + 1\cdot 5\cdot 1 + 5\cdot 1\cdot 1}{6^3} = \frac{15}{216}}$$

You win $\$3$ if all the dice match your number. Thus,

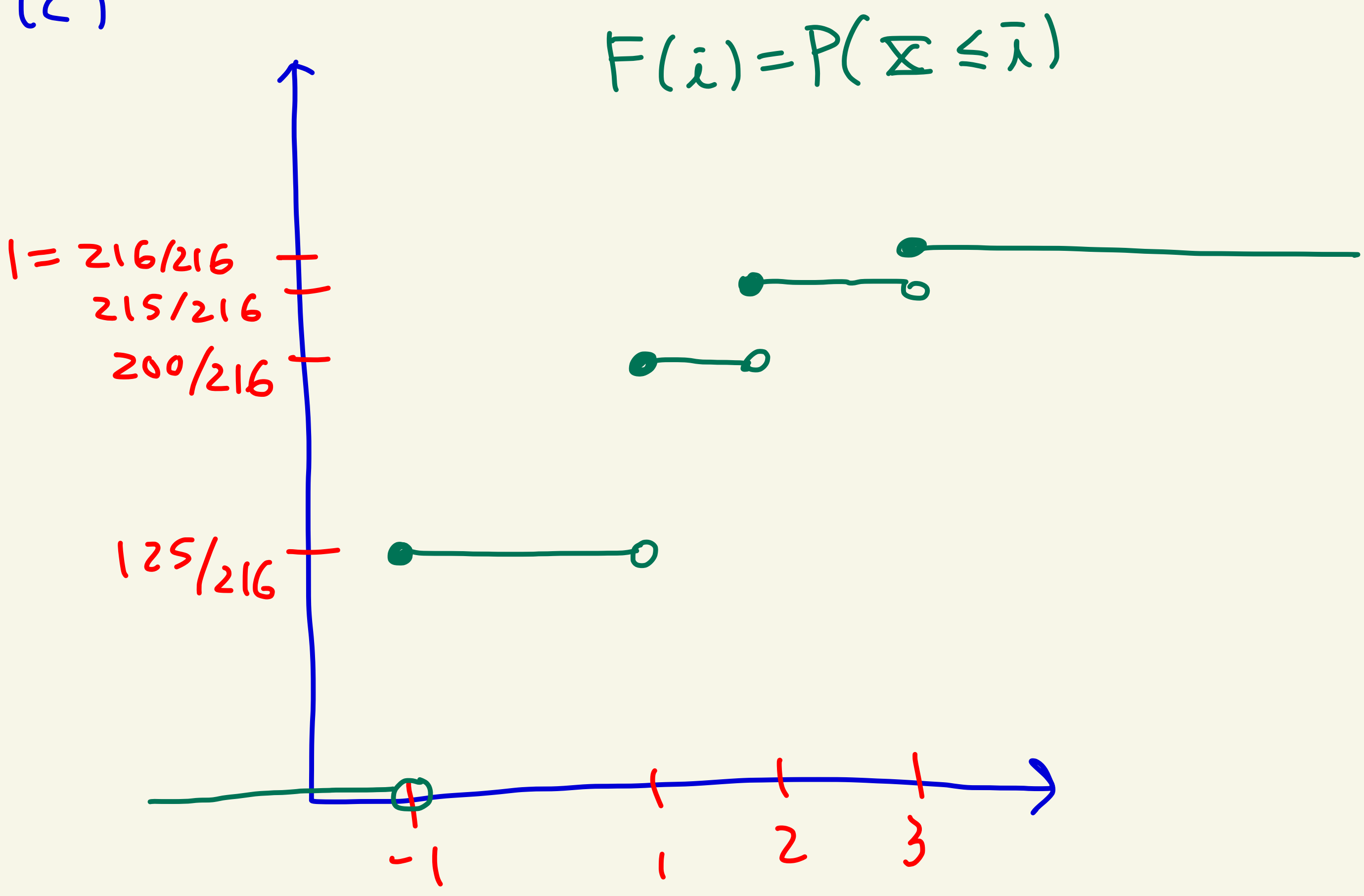
$$\boxed{p(3) = P(X=3) = \frac{1\cdot 1\cdot 1}{6^3} = \frac{1}{6^3} = \frac{1}{216}}$$

(b)



$p(i) = P(X = i)$
125/216
75/216
15/216
1/216
-1
1
2
3

(c)



$F(i) = P(X \le i)$
1 = 216/216
215/216
200/216
125/216
-1
1
2
3

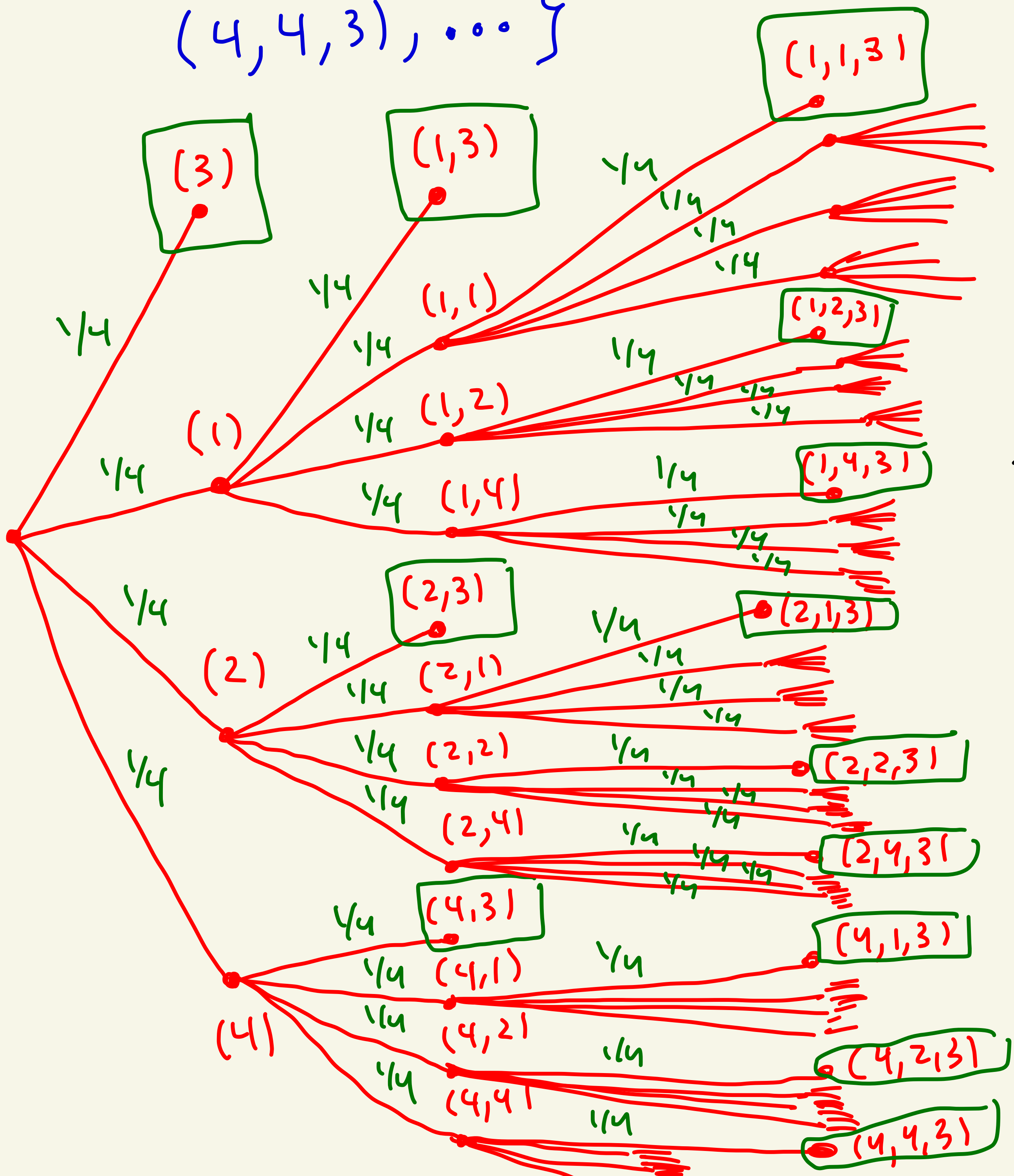
(d)

$$E[X] = (-\$1)\left(\frac{125}{216}\right) + (\$1)\left(\frac{75}{216}\right) + (\$2)\left(\frac{15}{216}\right) + (\$3)\left(\frac{1}{216}\right)$$

$$= -\$\frac{17}{216} \approx -\$0.0787$$

② (a)

$S = \{ (3), (1,3), (2,3), (4,3),$
$(1,1,3), (1,2,3), (1,4,3), (2,1,3),$
$(2,2,3), (2,4,3), (4,1,3), (4,2,3),$
$(4,4,3), \ldots \}$



The probability tree is on the left. The boxed elements are S. Each branch is probability 1/4. You multiply the probabilities

see next page ↓

We have

$$P(\{(3)\}) = \frac{1}{4}$$

$$P(\{(1,3)\}) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4^2}$$

$$P(\{(2,3)\}) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4^2}$$

$$P(\{(4,3)\}) = \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4^2}$$

$$P(\{(1,1,3)\}) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4^3}$$

$$P(\{(1,2,3)\}) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4^3}$$

$$P(\{(1,4,3)\}) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4^3}$$

$$P(\{(2,1,3)\}) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4^3}$$

$$P(\{(2,2,3)\}) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4^3}$$

$$P(\{(2,4,3)\}) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{4^3}$$

$$\vdots \qquad \vdots \qquad \vdots$$

and so on . . .

We check that P is a probability function by showing that the sum of P over S is 1. We have

$$\sum_{w \in S} P(\{w\}) = P(\{(3)\}) + P(\{(1,3)\}) + P(\{(2,3)\})$$
$$+ P(\{(4,3)\}) + P(\{(1,1,3)\})$$
$$+ P(\{(1,2,3)\}) + P(\{(1,4,3)\})$$
$$+ P(\{(2,1,3)\}) + P(\{(2,2,3)\})$$
$$+ P(\{(2,4,3)\}) + P(\{(4,1,3)\})$$
$$+ P(\{(4,2,3)\}) + P(\{(4,4,3)\})$$
$$+ \cdots$$
$$= \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^2} + \frac{1}{4^3} + \frac{1}{4^3}$$
$$+ \frac{1}{4^3} + \frac{1}{4^3} + \frac{1}{4^3} + \frac{1}{4^3} + \frac{1}{4^3} + \frac{1}{4^3}$$
$$+ \frac{1}{4^3} + \cdots$$
$$= \frac{1}{4} + 3 \cdot \frac{1}{4^2} + 3^2 \cdot \frac{1}{4^3} + 3^3 \cdot \frac{1}{4^4} + \cdots$$
$$= \frac{1}{4}\left[1 + \frac{3}{4} + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \left(\frac{3}{4}\right)^4 + \cdots\right]$$

$$= \frac{1}{4}\left[1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+\left(\frac{3}{4}\right)^3+\left(\frac{3}{4}\right)^4+\dots\right]$$

$$= \frac{1}{4}\cdot\left[\frac{1}{1-3/4}\right] = \frac{1}{4}\cdot\left[\frac{1}{1/4}\right] = 1$$

$$\boxed{1+x+x^2+x^3+\dots = \frac{1}{1-x} \quad \text{if } -1<x<1}$$

Thus, P is a probability function on the space S.

(b) $A = \{(1,1,3), (1,2,3), (1,4,3), (2,1,3), (2,2,3), (2,4,3), (4,1,3), (4,2,3), (4,4,3)\}$

$$P(A) = \underbrace{\frac{1}{4^3}+\frac{1}{4^3}+\dots+\frac{1}{4^3}}_{9 \text{ elements}} = 9\cdot\frac{1}{4^3} = \frac{9}{64} \approx 0.1406\dots \approx 14\%$$

(c) $B = \{(3), (1,3), (2,3), (4,3), (1,1,3), (1,2,3), (1,4,3), (2,1,3), (2,2,3), (2,4,3), (4,1,3), (4,2,3), (4,4,3)\}$

$$P(B) = \frac{1}{4} + 3 \cdot \frac{1}{4^2} + 9 \cdot \frac{1}{4^3} = \frac{16+12+9}{64}$$

$$= \frac{37}{64} \approx 0.578125\ldots \approx 57.8\%$$

(d) X = amount won or lost

$$E[X] = (\$5)\underbrace{\left(\frac{37}{64}\right)}_{\substack{\text{probability} \\ \text{3 is rolled} \\ \text{within first} \\ \text{3 rolls}}} + (-\$6)\underbrace{\left(\frac{27}{64}\right)}_{\substack{\text{probability} \\ \text{3 is rolled after} \\ \text{first 3 rolls}}}$$

$$= \frac{\$185 - \$162}{64} = \$\frac{23}{64} \approx \$0.359$$

If you can play the game many times then you'd expect to win on average \$0.36 per game. So good to play if you can play many times.

(3) $S = \{1, 2, 3, 4\}$

$P(\{1\}) = \frac{2}{8}$, $P(\{2\}) = \frac{1}{8}$

$P(\{3\}) = \frac{3}{8}$, $P(\{4\}) = \frac{2}{8}$

(a) Let X be the amount won or lost.

$$E[X] = (\$2)p(1) + (\$2)p(2) + (-\$1)p(3) + (-\$1)p(4)$$
$$= (\$2)\left(\tfrac{2}{8}\right) + (\$2)\left(\tfrac{1}{8}\right) + (-\$1)\left(\tfrac{3}{8}\right) + (-\$1)\left(\tfrac{2}{8}\right)$$
$$= \$\left(\tfrac{1}{8}\right) = \$0.125$$

(b)

Let a be the amount won or lost when 1 is rolled

Let b be the amount won or lost when 2 is rolled

Let c be the amount won or lost when 3 is rolled

Let d be the amount won or lost when 4 is rolled

Let Y be the amount won or lost.

Then

$$E[Y] = a\left(\tfrac{2}{8}\right) + b\left(\tfrac{1}{8}\right) + c\left(\tfrac{3}{8}\right) + d\left(\tfrac{2}{8}\right)$$
$$= \frac{2a + b + 3c + 2d}{8}$$

For $E[Y] = 0$ we would need

$$\boxed{2a + b + 3c + 2d = 0}.$$

There are infinitely many solutions to this equation. For example, you could have $a = \$1$, $b = \$1$, $c = -\$1$, $d = \$0$.

4

There are $47 = 52 - 5$ cards left in the deck.

(a) Out of the 47 remaining cards there are $13 - 3 = 10$ hearts left. The odds of getting 2 more hearts is

$$\frac{\binom{10}{2}}{\binom{47}{2}} = \frac{\left(\frac{10 \cdot 9}{2}\right)}{\left(\frac{47 \cdot 46}{2}\right)} = \boxed{\frac{45}{1081}} \approx \boxed{0.0416} \approx \boxed{4.16\%}$$

$$\boxed{\binom{n}{2} = \frac{n(n-1)}{2}}$$

(b) There are $13 - 3 = 10$ hearts left. There are $47 - 10 = 37$ non-hearts left.

$$\frac{\underbrace{\binom{10}{1}}_{\text{choose heart}} \underbrace{\binom{37}{1}}_{\text{choose non-heart}}}{\binom{47}{2}} = \frac{10 \cdot 37}{\left(\frac{47 \cdot 46}{2}\right)} = \boxed{\frac{370}{1081}} \approx \boxed{0.34} \approx \boxed{34\%}$$

(c) There are 3 queens left and $47-3=44$ non-queens left.

$$\frac{\overbrace{\binom{3}{1}}^{\substack{\text{choose}\\\text{a}\\\text{queen}}}\overbrace{\binom{44}{1}}^{\substack{\text{choose}\\\text{a}\\\text{non-queen}}}}{\binom{47}{2}} = \frac{3 \cdot 44}{1081} = \boxed{\frac{132}{1081}} \approx \boxed{0.122} \approx \boxed{12.2\%}$$

(d)

$$\frac{\overbrace{\binom{3}{2}}^{\text{choose 2 queens out of 3 queens left}}}{\binom{47}{2}} = \boxed{\frac{3}{1081}} \approx \boxed{0.00277521} \approx \boxed{0.28\%}$$

(e) This happens if you get exactly one queen or exactly 2 queens. Add (c) and (d) to get

$$\frac{132}{1081} + \frac{3}{1081} = \boxed{\frac{135}{1081}} \approx \boxed{0.12488...} \approx \boxed{12.49\%}$$

(f) Let X = amount won or lost

$E[X] =$

$$= (\$500)\left[\begin{array}{c}\text{probability}\\ \text{you get}\\ \text{a flush}\end{array}\right] + (-\$20)\left[\begin{array}{c}\text{probability you}\\ \text{don't get a}\\ \text{flush}\end{array}\right]$$

$$= (\$500)\underbrace{\left(\frac{45}{1081}\right)}_{\text{from part (a)}} + (-\$20)\left[1 - \frac{45}{1081}\right]$$

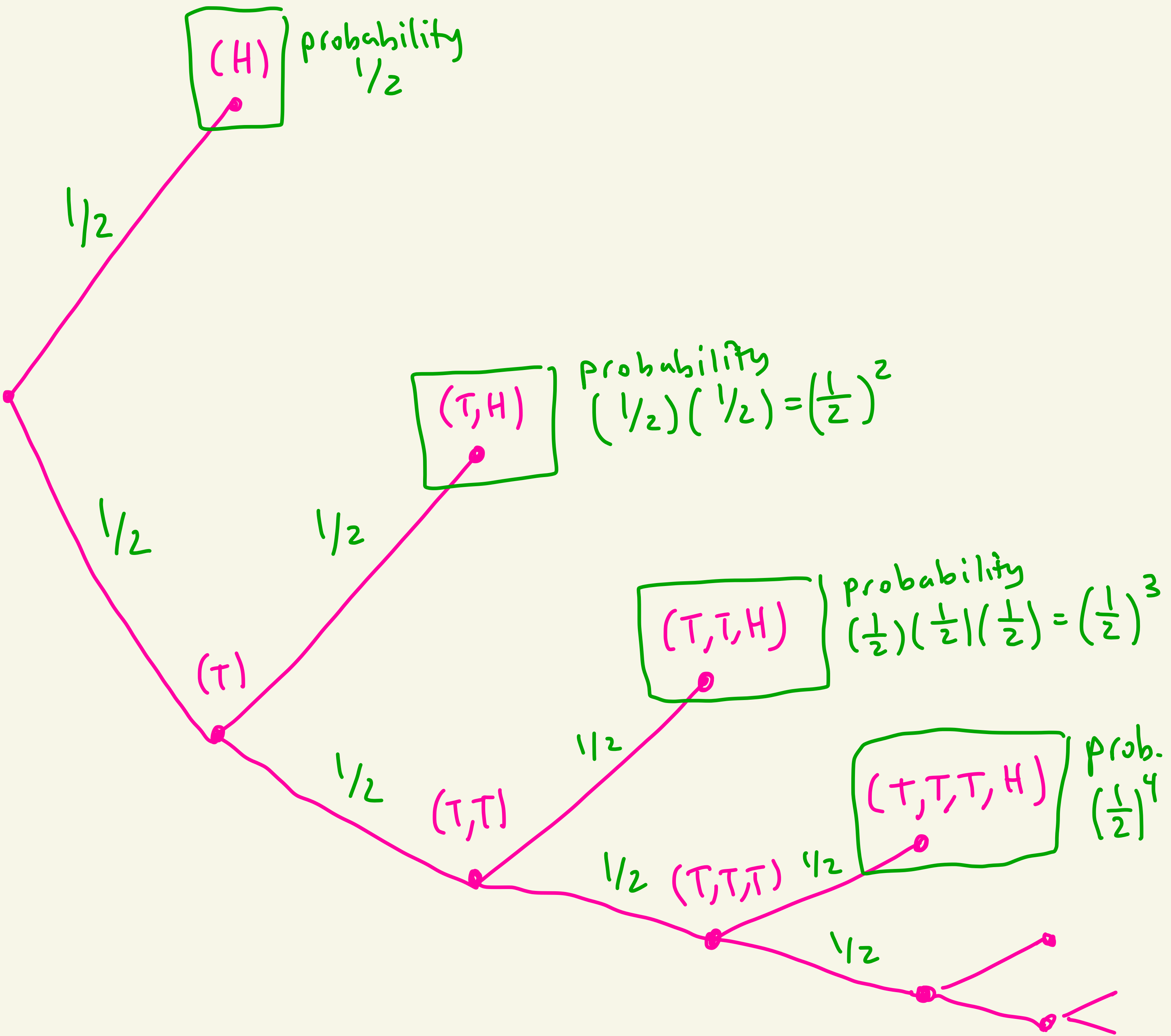
$$= \boxed{\$\frac{1780}{1081}} \approx \boxed{\$1.6466\ldots} \approx \boxed{\$1.65}$$

This is a good bet if you can play the game many times since on average over many plays you'd win \$1.65 per game.

⑤ From previous HW, the probability space is

$$S = \{(H), (T,H), (T,T,H), (T,T,T,H), (T,T,T,T,H), (T,T,T,T,T,H), \ldots\}$$

The probability tree looks like this:



Let

$E = \{(T,T,T,H), (T,T,T,T,H), (T,T,T,T,T,H), \ldots\}$

We want $P(E)$.

You could calculate this in two ways.

<u>Method 1</u>

$$P(E) = P(\{(T,T,T,H)\}) + P(\{(T,T,T,T,H)\}) + \ldots$$
$$= \left(\tfrac{1}{2}\right)^4 + \left(\tfrac{1}{2}\right)^5 + \left(\tfrac{1}{2}\right)^6 + \left(\tfrac{1}{2}\right)^7 + \ldots$$
$$= \left(\tfrac{1}{2}\right)^4 \left[1 + \tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3 + \ldots\right]$$
$$= \frac{1}{16}\left[\frac{1}{1-1/2}\right] = \frac{1}{16}\left[2\right] = \frac{1}{8}$$

$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$
if $-1 < x < 1$

<u>method 2</u>

$$P(E) = 1 - P(\overline{E}) = 1 - P(\{(H), (T,H), (T,T,H)\})$$
$$= 1 - \left[\tfrac{1}{2} + \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{2}\right)^3\right]$$
$$= 1 - \tfrac{1}{2} - \tfrac{1}{4} - \tfrac{1}{8} = \frac{8-4-2-1}{8}$$
$$= 1/8$$

Thus, $P(\overline{E}) = 1 - P(E) = 1 - \frac{1}{8} = \frac{7}{8}$

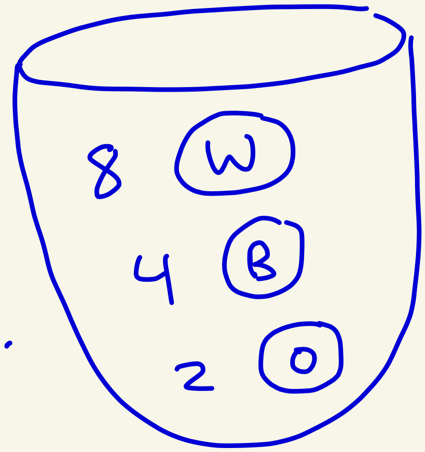
Let X be the amount won or lost.

Then

$$E[X] = (\$5)P(E) + (-\$1)P(\overline{E})$$
$$= (\$5)\left(\frac{1}{8}\right) + (-\$1)\left(\frac{7}{8}\right)$$
$$= -\$\frac{2}{8} = \boxed{-\$0.25}$$

The expected value is negative so in the long run if you did this bet many times you would expect to lose \$0.25 per bet.

6



Consider the following events:

WW – event two white balls are chosen.

WB – event one white ball and one black ball are chosen

WO – event one white ball and one orange ball are chosen

BB – event two black balls are chosen

BO – event one black ball and one orange ball are chosen.

OO – event two orange balls are chosen

Let X be the amount won or lost.

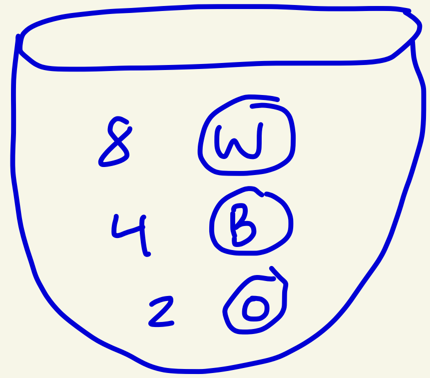
Note that the sample space of this experiment has size

$$\binom{14}{2} = \frac{14!}{2!\,12!} = \frac{14 \cdot 13 \cdot \cancel{12!}}{2!\,\cancel{12!}} = \frac{14 \cdot 13}{2} = 7 \cdot 13 = 91$$

Event	amount won or lost (value k of X)	probability of event $P(X=k)$ where k = amount won or lost
WW	-2	$\binom{8}{2} / \binom{14}{2} = \frac{28}{91}$
WO	-1	$\binom{8}{1}\binom{2}{1} / \binom{14}{2} = \frac{16}{91}$
OO	0	$\binom{2}{2} / \binom{14}{2} = \frac{1}{91}$
WB	1	$\binom{8}{1}\binom{4}{1} / \binom{14}{2} = \frac{32}{91}$
BO	2	$\binom{4}{1}\binom{2}{1} / \binom{14}{2} = \frac{8}{91}$
BB	4	$\binom{4}{2} / \binom{14}{2} = \frac{6}{91}$

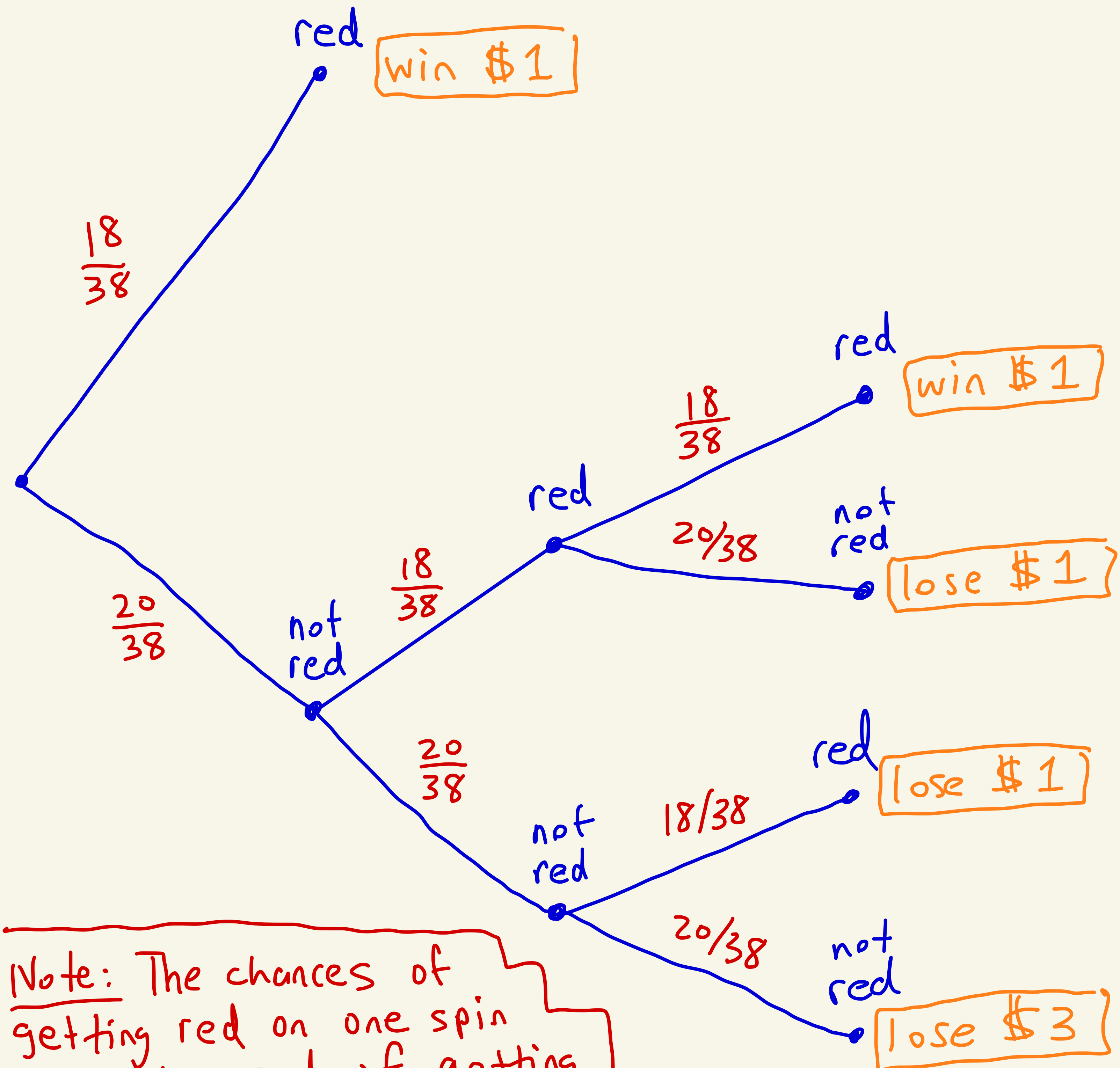
$$E[X] = (-2)\left(\frac{28}{91}\right) + (-1)\left(\frac{16}{91}\right) + (0)\left(\frac{1}{91}\right) + (1)\left(\frac{32}{91}\right) + (2)\left(\frac{8}{91}\right) + (4)\left(\frac{6}{91}\right)$$

$$= 0$$



⑦ Method 1 - Tree method

Let's draw the possibilities using a tree.



Note: The chances of getting red on one spin is 18/38 and of getting not red is 20/38

(a) Here $P(X>0)$ is $P(X=1)$ since we can only win \$1. Thus,

$$P(X>0) = \underbrace{\frac{18}{38}}_{\substack{\text{top}\\ \text{branch}\\ \text{of tree}\\ \text{leading to}\\ \text{win \$1}}} + \underbrace{\frac{20}{38}\cdot\frac{18}{38}\cdot\frac{18}{38}}_{\substack{\text{other branch}\\ \text{of tree}\\ \text{leading to}\\ \text{win \$1}}} = \boxed{\frac{4059}{6859}} \approx \boxed{0.5917\ldots}$$

(b) Do all the branches of the trees

$$E[X] = (\$1)\cdot\left[\frac{18}{38} + \frac{20}{38}\cdot\frac{18}{38}\cdot\frac{18}{38}\right]$$
$$+ (-\$1)\cdot\left[\frac{20}{38}\cdot\frac{18}{38}\cdot\frac{20}{38} + \frac{20}{38}\cdot\frac{20}{38}\cdot\frac{18}{38}\right]$$
$$+ (-\$3)\cdot\left[\frac{20}{38}\cdot\frac{20}{38}\cdot\frac{20}{38}\right]$$

$$= \boxed{-\$\frac{39}{361}} \approx \boxed{-\$0.11}$$

So even though the probability that we win is about 59%, we lose on average about \$0.11 per game.

7 **Method 2 - Formula Method**

Let R mean that a red number occured.
Let R' mean that a non-red number occured.

On one spin of the wheel there is an 18/38 chance that red will occur and a 20/38 chance that red will not occur.

Our sample space will be

$S = \{ R, R'RR, R'R'R, R'RR', R'R'R' \}$

where

R means we got red on first spin and stopped,
R' means got non-red on first spin,
$R'RR$ means got non-red on first spin, red on second spin, and red on third spin,
$R'R'R$ means got non-red on first and second spin and red on third spin,

and so on.

Since the spins are independent of each other we get that

$$P(R) = \frac{18}{38}$$
$$P(R'RR) = \frac{20}{38} \cdot \frac{18}{38} \cdot \frac{18}{38} = \frac{6480}{54872}$$

win $1

$$P(R'R'R) = \frac{20}{38} \cdot \frac{20}{38} \cdot \frac{18}{38} = \frac{7200}{54872}$$
$$P(R'RR') = \frac{20}{38} \cdot \frac{18}{38} \cdot \frac{20}{38} = \frac{7200}{54872}$$

lose $1

$$P(R'R'R') = \frac{20}{38} \cdot \frac{20}{38} \cdot \frac{20}{38} = \frac{8000}{54872}$$

lose $3

(a)

$$P(X>0) = \frac{18}{38} + \frac{6480}{54872} = \frac{4059}{6859} \approx 0.59177\overline{7}$$

(b)

$$E[X] = (1)\left(\frac{18}{38}\right) + (1)\left(\frac{6480}{54872}\right) + (-1)\left(\frac{7200}{54872}\right) + (-1)\left(\frac{7200}{54872}\right) + (-3)\left(\frac{8000}{54872}\right) = \frac{-39}{361} \approx -0.11$$

So on average you lose $0.11 per game even though you win ≈59% of the time. Therefore this is not a winning strategy.