

Homework 4 Solutions

①

(a) φ is not a ring homomorphism.

For example,

$$\varphi(2 \cdot 3) = \varphi(6) = 2 \cdot 6 = 12$$

but

$$\varphi(2) \cdot \varphi(3) = (2 \cdot 2)(2 \cdot 3) = 24.$$

~~Note:~~ φ is a group homomorphism

under $+$ because if $x, y \in \mathbb{Z}$ Then

$$\varphi(x+y) = 2(x+y) = 2x+2y = \varphi(x) + \varphi(y).$$

φ is not a ring homomorphism.

(b) φ satisfies $\varphi(AB) = \varphi(A)\varphi(B)$.

You can check this. However, it does not satisfy the additive property

$$\varphi(A+B) = \varphi(A) + \varphi(B). \text{ For example,}$$

$$\varphi\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \varphi\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

but

$$\varphi\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \varphi\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0 + 0 = 0$$

φ is NOT a ring homomorphism.

Let $a, b, c, d, e, f, g, h \in \mathbb{R}$, Then

(c) φ satisfies

$$\varphi \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right) = \varphi \left(\begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix} \right)$$

$$= \cancel{\varphi}(a+e) + (d+h)$$

$$= (a+d) + (e+h) = \varphi \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) + \varphi \left(\begin{pmatrix} e & f \\ g & h \end{pmatrix} \right).$$

However, φ does not satisfy the multiplicative property. For example,

$$\varphi \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) = \varphi \left(\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right) = 0$$

but

$$\varphi \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right) \varphi \left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right) = 1 \cdot 1 = 1,$$

(d) φ is a ring homomorphism.

Let $a, b, c, d \in \mathbb{Z}$. Then

$$\begin{aligned} \varphi((a, b) + (c, d)) &= \varphi(a+c, b+d) = a+c \\ &= \varphi(a, b) + \varphi(c, d) \end{aligned}$$

and

$$\varphi((a, b)(c, d)) = \varphi(ac, bd) = ac = \varphi(a, b)\varphi(c, d)$$

We have that

$$\begin{aligned}\ker(\varphi) &= \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid \varphi(a, b) = 0\} \\ &= \{(a, b) \in \mathbb{Z} \times \mathbb{Z} \mid a = 0\} \\ &\quad \text{\del{(0,0), (0,1), (0,-1), (0,2), (0,-2), ...)} \\ &= \{(0, b) \mid b \in \mathbb{Z}\} \\ &= \{\dots, (0, -2), (0, -1), (0, 0), (0, 1), (0, 2), \dots\}\end{aligned}$$

φ is not an isomorphism. It is onto,
but not 1-1. For example,

$$\begin{aligned}\varphi(1, 0) &= 1 \quad \text{and} \quad \varphi(1, 1) = 1 \\ \text{but } (1, 0) &\neq (1, 1).\end{aligned}$$

Suppose $\varphi: \mathbb{R} \rightarrow \mathbb{C}$ is a ring homomorphism.
Let's show that φ cannot be an isomorphism.
Note that $\varphi(1)^2 = \varphi(1)\varphi(1) = \varphi(1 \cdot 1) = \varphi(1)$,
so, $\varphi(1)[\varphi(1)-1] = 0$. Since our equation
is in \mathbb{C} , either $\varphi(1)=0$ or $\varphi(1)=1$.

If $\varphi(1)=0$, then φ is not $1\text{-}1$ since
 $\varphi(0)=0$ also.

If $\varphi(1)=1$, then $\varphi(-1) = -\varphi(1) = -1$.

If φ were an isomorphism, then $\exists r \in \mathbb{R}$
such that $\varphi(r) = i$. Then $\varphi(r)^2 = i^2 = -1$,

So, $\varphi(r^2) = -1$. But then, since
 φ is $1\text{-}1$, $r^2 = -1$. But this
can't happen since $r \in \mathbb{R}$.

So, in either case, φ is not an
isomorphism.

~~(\Rightarrow) If $a^2 - b^2 = (a+b)(a-b)$, then~~

$$a^2 - b^2 = (a+b)(a-b) = a^2 - ab + ba - b^2.$$

$$-a^2 + (a^2 - b^2) + b^2 = -a^2 + (a^2 - ab + ba - b^2) + b^2. \text{ Thus,}$$

$$0 = -ab + ba. \text{ Therefore, } ab = ba.$$

~~(\Leftarrow) (b) backwards in part (\Rightarrow).~~

③

$2\mathbb{Z} \neq 3\mathbb{Z}$ as rings: Suppose $\varphi: 2\mathbb{Z} \rightarrow 3\mathbb{Z}$

is a ring homomorphism. Consider $\varphi(2)$. What is it?

Note that $\varphi(2^2) = \varphi(2)\varphi(2) = \varphi(2)^2$ and

also $\varphi(2^2) = \varphi(4) = \varphi(2+2) = \varphi(2) + \varphi(2)$. So,

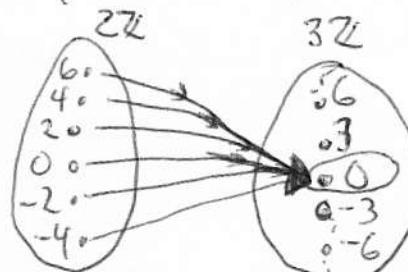
$\varphi(2)^2 = \varphi(2) + \varphi(2) = 2\varphi(2)$. So, $\varphi(2)[\varphi(2)-2] = 0$.

Since $3\mathbb{Z}$ is an integral domain, our equation gives that either $\varphi(2) = 0$ or $\varphi(2) = 2$. Since $2 \notin 3\mathbb{Z}$, we must have that $\varphi(2) = 0$. Then

if $2n \in 2\mathbb{Z}$, we have that

$$\varphi(2n) = \varphi(\underbrace{2+2+\dots+2}_{n \text{ times}}) = \varphi(2) + \dots + \varphi(2) = 0.$$

So our map looks like



That is,
 $\varphi(2n) = 0$
 $\forall n \in \mathbb{Z}$.
So φ can't be an isomorphism.

(4)

(a) We use the subring criteria. I will show that R_2 is a subring of $M_2(\mathbb{R})$. You show that R_1 is a subring of \mathbb{R} .

- $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \cdot 0 \\ 0 & 0 \end{pmatrix} \in R_2 \quad (\text{set } a=0, b=0)$

- Let $a, b, c, d \in \mathbb{Z}$. Then

$$\begin{pmatrix} a & 2b \\ b & a \end{pmatrix} - \begin{pmatrix} c & 2d \\ d & c \end{pmatrix} = \begin{pmatrix} (a-c) & 2(b-d) \\ (b-d) & (a-c) \end{pmatrix} \in R_2$$

and

$$\begin{aligned} \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} c & 2d \\ d & c \end{pmatrix} &= \begin{pmatrix} ac+2bd & 2ad+2bc \\ bc+ad & 2bd+ac \end{pmatrix} \\ &= \begin{pmatrix} ac+2bd & 2(ad+bc) \\ ad+bc & ac+2bd \end{pmatrix} \in R_2 \end{aligned}$$

So, R_2 is a subring of $M_2(\mathbb{R})$.

(b) Let $\varphi: R_1 \rightarrow R_2$ be given by

$$\varphi(a+b\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix},$$

Let $a, b, c, d \in \mathbb{Z}$. Then

$$\varphi((a+b\sqrt{2}) + (c+d\sqrt{2})) = \varphi((a+c) + (b+d)\sqrt{2}) = \begin{pmatrix} a+c & 2(b+d) \\ b+d & a+c \end{pmatrix}$$

and $\varphi(a+b\sqrt{2}) + \varphi(c+d\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} + \begin{pmatrix} c & 2d \\ d & c \end{pmatrix} = \begin{pmatrix} a+c & 2(b+d) \\ b+d & a+c \end{pmatrix}$

Thus φ satisfies the additive property of a ring homomorphism.

Also,

$$\begin{aligned}\varphi((a+b\sqrt{2})(c+d\sqrt{2})) &= \varphi((ac+2bd)+(ad+bc)\sqrt{2}) \\ &= \begin{pmatrix} ac+2bd & 2(ad+bc) \\ ad+bc & ac+2bd \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\varphi(a+b\sqrt{2})\varphi(c+d\sqrt{2}) &= \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \begin{pmatrix} c & 2d \\ d & c \end{pmatrix} \\ &= \begin{pmatrix} ac+2bd & 2(ad+bc) \\ ad+bc & ac+2bd \end{pmatrix}\end{aligned}$$

Thus, φ satisfies the multiplicative property of a ring homomorphism.

φ is 1-1:

Note that

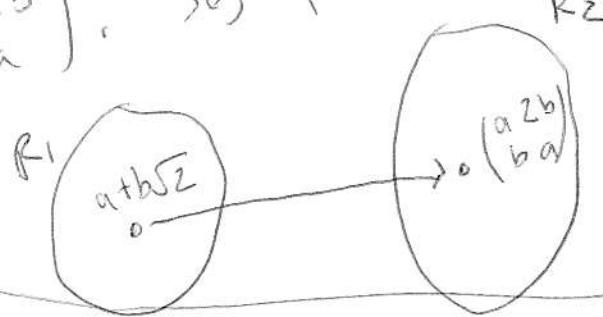
$$\begin{aligned}\ker(\varphi) &= \left\{ a+b\sqrt{2} \mid a, b \in \mathbb{Z} \text{ and } \varphi(a+b\sqrt{2}) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \\ &= \left\{ a+b\sqrt{2} \mid a, b \in \mathbb{Z} \text{ and } \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} \\ &= \left\{ 0+0\sqrt{2} \right\} = \{0\},\end{aligned}$$

Hence φ is 1-1 by a theorem from class.

φ is onto: Let $M \in R_2$. Then $M = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$ for some $a, b \in \mathbb{Z}$. Then $a+b\sqrt{2} \in R_1$ and $\varphi(a+b\sqrt{2}) = M$. So, φ is onto.

$$\varphi(a+b\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}. \text{ So, } \varphi \text{ is onto.}$$

Thus φ is an isomorphism.
So, R_1 and R_2 are isomorphic.



~~Section 18 Solutions~~



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~~Then, ϕ is an isomorphism, so R and R_2 are isomorphic as rings.~~

⑤ (continued...) By a thm in class, $\varphi(0)=0$.

If $n < 0$, then by the previous page

$$\varphi(n) = \varphi(-n) = \text{[redacted]} \begin{cases} 0 & \text{if } \varphi(1)=0 \\ -n & \text{if } \varphi(1)=1 \end{cases}$$

Hence, if $\varphi(1)=0$, then $\varphi(x)=0$ for all $x \in \mathbb{Z}$. If $\varphi(1)=1$, then $\varphi(x)=x$ for all $x \in \mathbb{Z}$. So, there are two ring homomorphisms $\varphi: \mathbb{Z} \rightarrow \mathbb{Z}$.

(6)

(a) Certainly

 ~~$(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}) \in R$~~ . (Just set $a=0$ and $b=0$.)
Let $a, b, c, d \in \mathbb{Z}$. Then

$$\left(\begin{array}{cc} a & 0 \\ 0 & b \end{array}\right) - \left(\begin{array}{cc} c & 0 \\ 0 & d \end{array}\right) = \left(\begin{array}{cc} a-c & 0 \\ 0 & b-d \end{array}\right) \in R$$

and

$$\left(\begin{array}{cc} a & 0 \\ 0 & b \end{array}\right) \left(\begin{array}{cc} c & 0 \\ 0 & d \end{array}\right) = \left(\begin{array}{cc} ac & 0 \\ 0 & bd \end{array}\right) \in R.$$

By the subring criteria, R is a subring of $M_2(\mathbb{R})$.(b) Define the function $\varphi: R \rightarrow \mathbb{Z} \times \mathbb{Z}$

$$\text{by } \varphi \left(\begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) = (a, b),$$

 φ is a ring homomorphism: Let $a, b, c, d \in \mathbb{Z}$,

$$\begin{aligned} \text{Then } \varphi \left(\left(\begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) + \left(\begin{array}{cc} c & 0 \\ 0 & d \end{array} \right) \right) &= \varphi \left(\begin{array}{cc} a+c & 0 \\ 0 & b+d \end{array} \right) = (a+c, b+d) \\ &= (a, b) + (c, d) = \varphi \left(\begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) + \varphi \left(\begin{array}{cc} c & 0 \\ 0 & d \end{array} \right) \end{aligned}$$

and

$$\begin{aligned} \varphi \left(\left(\begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) \circ \left(\begin{array}{cc} c & 0 \\ 0 & d \end{array} \right) \right) &= \varphi \left(\begin{array}{cc} ac & 0 \\ 0 & bd \end{array} \right) = (ac, bd) \\ &= (a, b)(c, d) = \varphi \left(\begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) \circ \varphi \left(\begin{array}{cc} c & 0 \\ 0 & d \end{array} \right) \end{aligned}$$

φ is 1-1: Let $a, b, c, d \in \mathbb{Z}$, suppose $\varphi \left(\begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) = \varphi \left(\begin{array}{cc} c & 0 \\ 0 & d \end{array} \right)$.

Then $(a, b) = (c, d)$. Hence $a = c$ and $b = d$.

Thus, $\left(\begin{array}{cc} a & 0 \\ 0 & b \end{array} \right) = \left(\begin{array}{cc} c & 0 \\ 0 & d \end{array} \right)$.

φ is onto: Let $(a, b) \in \mathbb{Z} \times \mathbb{Z}$. Then
 $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in R$ and $\varphi\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\right) = (a, b)$. Thus

φ is onto.

φ is an isomorphism. Thus, R and $\mathbb{Z} \times \mathbb{Z}$
are isomorphic as rings.

(7)

We have that

$$(a) \varphi(0) = \varphi(0+0) = \varphi(0) + \varphi(0),$$

Thus,

$$\underbrace{-\varphi(0)}_{0'} + \varphi(0) = \underbrace{-\varphi(0)}_{0'} + \varphi(0) + \varphi(0)$$

$$\text{So, } 0' = \varphi(0).$$

(b) The statement $-\varphi(a) = \varphi(-a)$ means that $\varphi(-a)$ is the additive inverse for $\varphi(a)$. This can be shown by showing that

$$\varphi(a) + \varphi(-a) = \varphi(-a) + \varphi(a) = 0'.$$

Indeed, we have that

$$\varphi(a) + \varphi(-a) = \varphi(a + (-a)) = \varphi(0) = 0'$$

and

$$\varphi(-a) + \varphi(a) = \varphi((-a) + a) = \varphi(0) = 0'.$$

(c) ~~We use the subring criteria.~~ We use the subring criteria.

Note that $0' = \varphi(0)$ and $0 \in S$.

Thus, $0' \in \varphi(S) = \{\varphi(x) \mid x \in S\}$.

Let $a, b \in \varphi(S)$. Then $a = \varphi(x)$

and $b = \varphi(y)$ where $x, y \in S$.

Note that $x-y \in S$ and $x-y \in S$ since S is a subring of R .

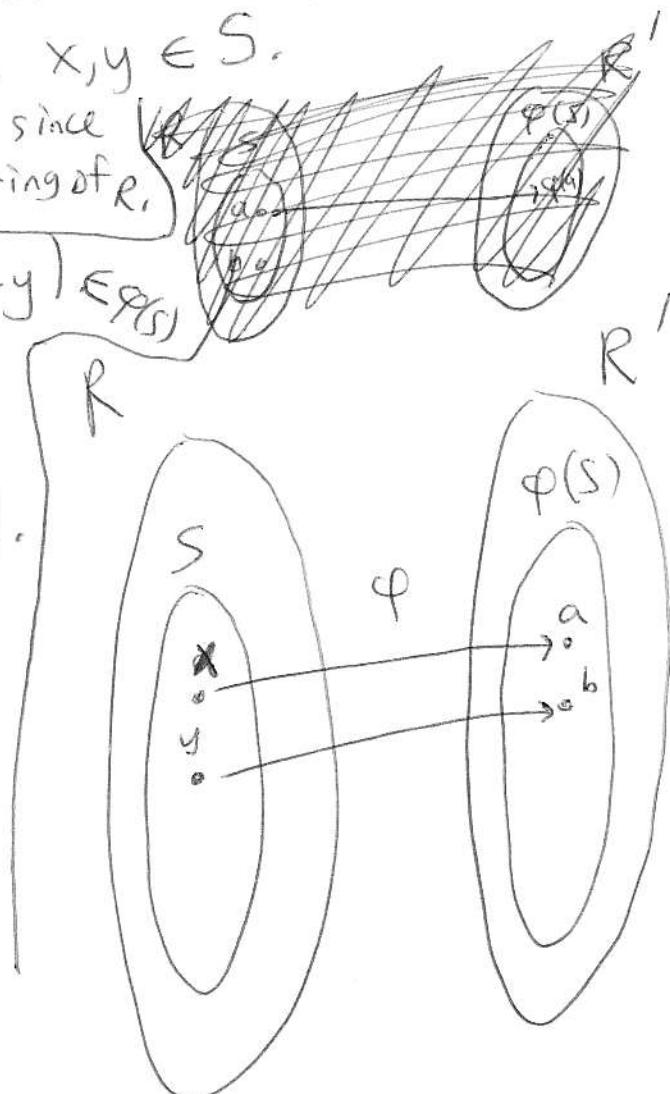
Then

$$a-b = \varphi(x) - \varphi(y) = \varphi(x-y) \in \varphi(S)$$

~~and~~

$$ab = \varphi(x)\varphi(y) = \varphi(xy) \in \varphi(S).$$

So, $\varphi(S)$ is a subring
of R' .



(d) Let $a \in \varphi(R)$, Then
 $a = \varphi(x)$ where $x \in R$,

Then,

$$\varphi(1) \cdot a = \varphi(1) \cdot \varphi(x) = \varphi(1 \cdot x) = \varphi(x) = a$$

and

$$a \cdot \varphi(1) = \varphi(x) \cdot \varphi(1) = \varphi(x \cdot 1) = \varphi(x) = a.$$

Thus, $\varphi(1)$ is a multiplicative identity for $\varphi(R)$.

