Math 456
Homework # 4 - Ring homomorphisms

1. For each part do the following: (a) Are the following functions ϕ ring homomorphisms? If so, prove it. If not, disprove it. (b) If you prove that ϕ is a homomorphism then also find the kernel of ϕ. (c) Is ϕ an isomorphism?

   (a) $\phi : \mathbb{Z} \to 2\mathbb{Z}$ given by $\phi(x) = 2x$.

   (b) $\phi : M_2(\mathbb{R}) \to \mathbb{R}$ given by $\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = ad - bc$.

   (c) $\phi : M_2(\mathbb{R}) \to \mathbb{R}$ given by $\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) = a + d$.

   (d) $\phi : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $\phi(a, b) = a$.

2. Show that $\mathbb{R}$ and $\mathbb{C}$ are not isomorphic as rings.

3. Show that $2\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic as rings.

4. Let
$$R_1 = \{a + b\sqrt{2} \mid a, b \in \mathbb{Z}\}$$
and
$$R_2 = \left\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}.$$

   (a) Show that R_1 is a subring of $\mathbb{R}$ and R_2 is a subring of $M_2(\mathbb{R})$.

   (b) Show that $\phi : R_1 \to R_2$ given by $\phi(a + b\sqrt{2}) = \begin{pmatrix} a & 2b \\ b & a \end{pmatrix}$ is an isomorphism of rings.

5. Find all ring homomorphisms $\phi : \mathbb{Z} \to \mathbb{Z}$.

6. Let
$$R = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbb{Z} \right\}.$$

(a) Prove that R is a subring of $M_2(\mathbb{R})$.

(b) Prove that R is isomorphic to $\mathbb{Z} \times \mathbb{Z}$.

7. Let R and R' be rings. Let $\phi : R \to R'$ be a ring homomorphism. Prove the following:

(a) Let 0 and $0'$ be the additive identities of R and R'. Prove that $\phi(0) = 0'$.

(b) Let $a \in R$. Then $-\phi(a) = \phi(-a)$.

(c) If S is a subring of R then

$$\phi(S) = \{\phi(x) \mid x \in S\}$$

is a subring of R'.

(d) If R has a multiplicative identity denoted by 1, then $\phi(1)$ is a multiplicative identity of $\phi(R)$.