Math 5680

Homework # 2

Sequences of functions, series of functions, Weierstrass M-Test, Analytic convergence theorem

---

1. Let $A = \mathbb{R} \subseteq \mathbb{C}$ and $f_n : A \to \mathbb{C}$ be defined for $n \geq 2$ as follows

$$f_n(x) = \begin{cases} -1 & \text{for } x \leq -1/n \\ nx & \text{for } -1/n < x < 1/n \\ 1 & \text{for } 1/n \leq x \end{cases}$$

   (a) Draw a picture of f_n for $n = 2$ and $n = 3$ and $n = 4$.

   (b) Let

$$f(x) = \begin{cases} -1 & \text{for } x < 0 \\ 0 & \text{for } x = 0 \\ 1 & \text{for } 0 < x \end{cases}$$

   Prove that f_n converges pointwise to f on $A = \mathbb{R}$.

---

2. Consider the sequence of functions $\{f_n\}_{n=1}^{\infty}$ where

$$f_n(z) = \frac{z^3}{n^2}$$

   Let $r > 0$ be fixed. Show that $\{f_n\}$ converges uniformly to the zero function on the disc $D(0; r)$ of radius r centered at 0.

---

3. Let

$$g(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}$$

   Given a positive real number r, define the sets

$$A_r = \{z \mid |z| \leq r\}$$

   (a) Draw a picture of A_1 and A_π.

   (b) If $0 \leq r < 1$ prove that that the series $g(z)$ converges absolutely and uniformly on A_r.

4. Let
$$g(z) = \sum_{n=1}^{\infty} \frac{1}{z^n}$$
and $A = \{z \mid |z| > 1\}$.
   (a) Show that $g(z)$ is analytic on A.
   (b) Find the derivative of $g(z)$ on A.

5. Let
$$g(z) = \sum_{n=1}^{\infty} \frac{1}{n!\, z^n}$$
   (a) Show that g is analytic on $A = \mathbb{C} - \{0\}$.
   (b) Find the derivative of $g(z)$ on A.

6. Suppose that $\sum_{k=1}^{\infty} g_k(z)$ converges uniformly on some subset A of $\mathbb{C}$. Prove that the sequence (g_k) converges uniformly to the zero function f_0 on A. Here $f_0(z) = 0$ for all $z \in A$.

THE NEXT PROBLEM ISN'T NECESSARY TO DO.

We used the fact given below in the proof of the analytic convergence theorem. Do the problem if you want an extra challenge and you are familiar with theorems about compactness.

A. Let $A \subseteq \mathbb{C}$ be an open set and $z_0 \in A$. Let $r > 0$. Suppose that
$$B = \{z \mid \ |z - z_0| \leq r\}$$
is contained in A. Prove that there is a number $\rho > r$ such that the circle γ of radius ρ centered at z_0 satisfies (i) γ is contained in A and (ii) γ contains B.

[**Hint:** The boundary of B is a compact subset of $\mathbb{C}$. For each point on the boundary of B pick a open disk that surrounds that point and is contained in A. Then shrink those disks in half. Then use compactness to get a finite sub-cover.]