Math 4300 - Homework # 2

Metric Geometries

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1. In the Euclidean plane, find the coordinates of the following points on the given line using the standard ruler. Draw a picture of the standard ruler function.

   (a) line: L_{-2}
   points: $(-2,-3)$, $(-2,-2)$, $(-2,-3/2)$, $(-2,0)$, $(-2,1)$, $(-2,\pi)$

   (b) line: $L_{-2,4}$
   points: $(-2,-3)$, $(-2,-2)$, $(-2,-3/2)$, $(-2,0)$, $(-2,1)$, $(-2,\pi)$

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2. In the Hyperbolic plane, find the coordinates of the following points on the given line using the standard ruler. Draw a picture of the standard ruler function.

   (a) line: ${}_2L$
   points: $(2,0.0001)$, $(2,0.4)$, $(2,1)$, $(2,e)$, $(2,5)$, $(2,10)$

   (b) line: ${}_1L_{\sqrt{10}}$
   points:
   $(-2.16,0.12)$, $\left(-1,\sqrt{6}\right)$, $(0,3)$, $(1,\sqrt{10})$, $(2,3)$, $(3,\sqrt{6})$, $(4.16,0.12)$
   (Some of the above points are approximations.)

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3. In the Euclidean plane, find the distance between the given points.

   (a) $P=(1,2)$ and $Q=(3,4)$

   (b) $P=(-3,1)$ and $Q=(5,10)$

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4. In the Hyperbolic plane, find the distance between the given points.

   (a) $P=(1,2)$ and $Q=(5,6)$

   (b) $P=(6,\pi^2)$ and $Q=(6,2)$

5. In the Euclidean plane, find a point P on the line $L_{3,-3}$ with coordinate -2 using the standard ruler.

6. In the Euclidean plane, find a ruler f for the line $\overleftrightarrow{PQ}$ where $f(P) = 0$ and $f(Q) > 0$.
   (a) $P = (2, 3)$ and $Q = (2, 5)$
   (b) $P = (2, 3)$ and $Q = (2, -5)$
   (c) $P = (2, 3)$ and $Q = (4, 0)$

7. In the Hyperbolic plane, find a ruler f for the line $\overleftrightarrow{PQ}$ where $f(P) = 0$ and $f(Q) > 0$.
   (a) $P = (2, 3)$ and $Q = (2, 1/3)$
   (b) $P = (2, 3)$ and $Q = (-1, 6)$

8. Let $(\mathscr{P}, \mathscr{L}, d)$ be an metric geometry. Let $P \in \mathscr{P}$ and let ℓ be a line through P. Let $r > 0$ be a real number. Prove there is a point $Q \in \mathscr{P}$ with $Q \in \ell$ and $d(P, Q) = r$.

9. Prove that a line in a metric geometry has infinitely many points.

10. Recall from class that

$$\sinh(t) = \frac{e^t - e^{-t}}{2} \qquad \cosh(t) = \frac{e^t + e^{-t}}{2}$$

$$\tanh(t) = \frac{\sinh(t)}{\cosh(t)} \qquad \operatorname{sech}(t) = \frac{1}{\cosh(t)}$$

Prove the following are true for all $t \in \mathbb{R}$.

   (a) $(\cosh(t))^2 - (\sinh(t))^2 = 1$

(b) $\cosh(t) > 0$

(c) $(\tanh(t))^2 + (\operatorname{sech}(t))^2 = 1$

(d) $\operatorname{sech}(t) > 0$

(e) Prove that $\tanh(t)$ is a strictly increasing function.
That is, if $t_1 < t_2$, then $\tanh(t_1) < \tanh(t_2)$.
