Math 4300 - Homework # 1

Abstract and Incidence Geometries

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1. In the Euclidean plane $\mathscr{E} = (\mathbb{R}^2, \mathscr{L}_E)$, find the line through P and Q and draw a picture where
   (a) $P = (-1, 2)$ and $Q = (3, 2)$
   (b) $P = (-4, -\sqrt{2})$ and $Q = (-4, 2)$
   (c) $P = (2, 1)$ and $Q = (4, 3)$

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2. In the Hyperbolic plane $\mathscr{H} = (\mathbb{H}, \mathscr{L}_H)$, find the line through P and Q and draw a picture where
   (a) $P = (1, 2)$ and $Q = (3, 4)$
   (b) $P = (\pi, \sqrt{2})$ and $Q = (\pi, 2)$
   (c) $P = (2, 1)$ and $Q = (4, 3)$

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3. In the Euclidean plane $\mathscr{E} = (\mathbb{R}^2, \mathscr{L}_E)$, determine if the points are collinear or non-collinear.
   (a) $P = (3, 2)$, $Q = (3, 1)$, $R = (1, -1)$
   (b) $P = (2, 1)$, $Q = (4, 3)$, $R = (6, 5)$
   (c) $P = (0, 1)$, $Q = (0, 3)$, $R = (0, -5)$, $S = (0, 10)$

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4. In the Hyperbolic plane $\mathscr{H} = (\mathbb{H}, \mathscr{L}_H)$ determine if the points are collinear or non-collinear.
   (a) $A = (-2, 2)$, $B = (-2, 4)$, and $C = (-2, 300)$
   (b) $P = (0, 1)$, $Q = (1, 2)$, and $R = (4, 1)$
   (c) $A = (1, 1)$, $B = (3, 1)$, and $C = (2, 3)$

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5. In the Euclidean plane $\mathscr{E} = (\mathbb{R}^2, \mathscr{L}_E)$, determine if the lines are parallel or not.

(a) L_1 and L_1

(b) L_{-3} and L_1

(c) L_{-3} and $L_{1,1}$

(d) $L_{-1,2}$ and $L_{1,1}$

(e) $L_{3,2}$ and $L_{3,-1}$

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6. In the Hyperbolic plane $\mathscr{H} = (\mathbb{H}, \mathscr{L}_H)$ determine if the lines are parallel or not.

   (a) ${}_0L_1$ and ${}_5L_2$

   (b) ${}_0L_1$ and ${}_2L_2$

   (c) ${}_0L_{10}$ and ${}_5L_2$

   (d) ${}_0L_{10}$ and ${}_0L_{10}$

   (e) ${}_1L_{10}$ and ${}_{-5}L$

   (f) ${}_1L_1$ and ${}_2L_2$

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7. Let $(\mathscr{P}, \mathscr{L})$ be an incidence geometry. Suppose that P, Q, R are distinct points from $\mathscr{P}$ and that they are collinear. Prove that there is a unique line from $\mathscr{L}$ that passes through all three points.

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8. Let $(\mathscr{P}, \mathscr{L})$ be an incidence geometry. Let ℓ be a line. Prove that there must exist a point P such that P does not lie on ℓ.

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9. Let $(\mathscr{P}, \mathscr{L})$ be an incidence geometry. Let P be any point. Prove that there exists at least one line ℓ such that P does not lie on ℓ.

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10. Let $(\mathscr{P}, \mathscr{L})$ be an incidence geometry. Let P be any point. Prove that there exist points Q and R such that P, Q, and R are non-collinear.

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11. (a) In the Euclidean plane $\mathscr{E} = (\mathbb{R}^2, \mathscr{L}_E)$, find all the lines through the point $P = (0, 1)$ that are parallel to the line L_6.

(b) Consider the Euclidean plane $\mathscr{E} = (\mathbb{R}^2, \mathscr{L}_E)$. Let ℓ be a line in $\mathscr{L}_E$ and P be a point in $\mathbb{R}^2$ that does not lie on ℓ. Prove that there exists a unique line m such that P lies on m and m is parallel to ℓ

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12. (a) In the Hyperbolic plane $\mathscr{H} = (\mathbb{H}, \mathscr{L}_H)$, find an infinite number of lines through the point $(0, 1)$ that are parallel to the line ${}_6L$.

    (b) Conclude that the statement about parallel lines given in problem 11(b) above is not true in the hyperbolic plane $\mathscr{H}$.

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13. Consider the hyperbolic plane $\mathscr{H} = (\mathbb{H}, \mathscr{L}_H)$. Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ where $x_1 \neq x_2$. Prove that P and Q both lie on ${}_cL_r$ where

$$c = \frac{y_2^2 - y_1^2 + x_2^2 - x_1^2}{2(x_2 - x_1)} \qquad \text{and} \qquad r = \sqrt{(x_1 - c)^2 + y_1^2}$$

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14. Prove that the hyperbolic plane $\mathscr{H} = (\mathbb{H}, \mathscr{L}_H)$ is an incidence geometry.
