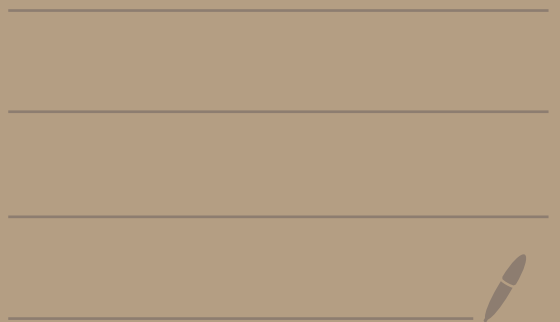


Math 4740

Hw 1 Solutions



$$\textcircled{1}(a) S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$$

$P(\{(a,b)\}) = \frac{1}{36}$ for any (a,b) in S . That is, each element of S is equally weighted. If E is an event then $P(E) = \frac{|E|}{|S|} = \frac{|E|}{36}$

$$\textcircled{1}(b) A = \{(1,1)\}$$

$$B = \{(1,3), (2,2), (3,1)\}$$

$$\textcircled{1}(c) A \cup B = \{(1,1), (1,3), (2,2), (3,1)\}$$

$$A \cap B = \emptyset$$

$$\overline{A} = \{ \text{all elements of } S \text{ except } (1,1) \}$$

$$= \{ (1,2), (1,3), (1,4), (1,5), (1,6), \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$$

$$\begin{aligned}\overline{B} &= \{ \text{all elements of } S \text{ except } (1,3), (2,2), (3,1) \} \\ &= \{ (1,1), (1,2), (1,4), (1,5), (1,6), \\ &\quad (2,1), (2,3), (2,4), (2,5), (2,6), \\ &\quad (3,2), (3,3), (3,4), (3,5), (3,6), \\ &\quad (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\ &\quad (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\ &\quad (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}\end{aligned}$$

①(d)

$A \cup B$ consists of all the rolls where the sum of the dice is either 2 or 4

$A \cap B$ consists of all the rolls where the sum of the dice is 2 and 4 at the same time which is impossible, hence $A \cap B$ is empty

$\overline{A}$ consists of all the rolls of the dice where the sum is not 2

$\overline{B}$ consists of all the rolls of the dice where the sum is not 4

① (e)

$$P(A) = \frac{1}{36}$$

$$P(B) = \frac{3}{36}$$

$$P(A \cup B) = \frac{4}{36}$$

$$P(A \cap B) = 0$$

$$P(\overline{A}) = \frac{35}{36}$$

$$P(\overline{B}) = \frac{33}{36}$$

② (a)

$$S = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$$

$P(\{a,b\}) = \frac{1}{36}$ for any (a,b) in S . That is, each element of S is equally weighted. If E is an event then $P(E) = \frac{|E|}{|S|} = \frac{|E|}{36}$

② (b)

$$A = \{ (1,2), (1,4), (1,6), \\ (2,1), (2,2), (2,3), (2,4), (2,5), (2,6), \\ (3,2), (3,4), (3,6), \\ (4,1), (4,2), (4,3), (4,4), (4,5), (4,6), \\ (5,2), (5,4), (5,6), \\ (6,1), (6,2), (6,3), (6,4), (6,5), (6,6) \}$$

$$B = \{ (2,2), (2,4), (2,6), (4,2), (4,4), (4,6), \\ (6,2), (6,4), (6,6) \}$$

$$C = \{ (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), \\ (2,1), (2,3), (2,5), \\ (3,1), (3,2), (3,3), (3,4), (3,5), (3,6), \\ (4,1), (4,3), (4,5), \\ (5,1), (5,2), (5,3), (5,4), (5,5), (5,6), \\ (6,1), (6,3), (6,5) \}$$

$$D = \{ (1,1), (1,3), (1,5), (3,1), (3,3), (3,5), \\ (5,1), (5,3), (5,5) \}$$

② (c)

$$A \cap C = \{ (1,2), (1,4), (1,6), (2,1), (2,3), (2,5), \\ (3,2), (3,4), (3,6), (4,1), (4,3), (4,5), \\ (5,2), (5,4), (5,6), (6,1), (6,3), (6,5) \}$$

$$\bar{A} = D$$

$$B \cap D = \emptyset$$

$$B \cup D = \{ (1,1), (1,3), (1,5), (2,2), (2,4), (2,6), \\ (3,1), (3,3), (3,5), (4,2), (4,4), (4,6), \\ (5,1), (5,3), (5,5), (6,2), (6,4), (6,6) \}$$

$$\bar{B} = C$$

$$\bar{D} = A$$

②(d)

$A \cap C$ are the dice rolls where one of the dice is even and one of the dice is odd.
 $\bar{A}$ are the dice rolls where both dice are odd.

$B \cup D$ are the dice rolls where either both dice are even or both dice are odd.

$B \cap D$ is empty since it consists of the dice rolls where both dice are even and both dice are odd, which can't happen

②(e)

$$\begin{aligned} P(A) &= \frac{27}{36} & P(B) &= \frac{9}{36} & P(C) &= \frac{27}{36} \\ P(D) &= \frac{9}{36} & P(A \cap C) &= \frac{18}{36} & P(\bar{A}) &= \frac{9}{36} \\ P(B \cap D) &= 0 & P(B \cup D) &= \frac{18}{36} & P(\bar{B}) &= \frac{27}{36} \\ P(\bar{D}) &= \frac{27}{36} \end{aligned}$$

③(a)

$$S = \{ (H, H, H, H), (H, H, H, T), (H, H, T, H), (H, H, T, T), \\ (H, T, H, H), (H, T, H, T), (H, T, T, H), (H, T, T, T), \\ (T, H, H, H), (T, H, H, T), (T, H, T, H), (T, H, T, T), \\ (T, T, H, H), (T, T, H, T), (T, T, T, H), (T, T, T, T) \}$$

$$P(\{ (a, b, c, d) \}) = \frac{1}{16} \text{ for any } (a, b, c, d) \text{ in } S.$$

That is the elements are equally weighted.

If E is an event then

$$P(E) = \frac{|E|}{|S|} = \frac{|E|}{16}$$

③(b)

$$A = \{ (H, H, H, H), (H, H, H, T), (H, H, T, H), (H, H, T, T), \\ (H, T, H, H), (H, T, H, T), (H, T, T, H), (H, T, T, T) \}$$

$$B = \{ (H, T, H, T), (H, T, T, T), (T, T, H, T), (T, T, T, T) \}$$

③(c)

$$A \cup B = \{ (H, H, H, H), (H, H, H, T), (H, H, T, H), (H, H, T, T), \\ (H, T, H, H), (H, T, H, T), (H, T, T, H), (H, T, T, T), \\ (T, T, H, T), (T, T, T, T) \}$$

$$A \cap B = \{ (H, T, H, T), (H, T, T, T) \}$$

$$\overline{A} = \{ (T, H, H, H), (T, H, H, T), (T, H, T, H), (T, H, T, T), \\ (T, T, H, H), (T, T, H, T), (T, T, T, H), (T, T, T, T) \}$$

$$\overline{B} = \{ (H, H, H, H), (H, H, H, T), (H, H, T, H), (H, H, T, T), \\ (H, T, H, H), (H, T, T, H), \\ (T, H, H, H), (T, H, H, T), (T, H, T, H), (T, H, T, T), \\ (T, T, H, H), (T, T, T, H) \}$$

③(d)

$A \cup B$ are the elements that either begin with a head on the first flip OR have a tails on the second or fourth flip.

$A \cap B$ are the elements with heads on the first flip and tails on the second and fourth flips

$\overline{A}$ are the elements that have a tails on the first flip.

$\overline{B}$ are the elements where either the second or the fourth flip is a heads.

③ (e)

$$P(A) = \frac{8}{16}$$

$$P(B) = \frac{4}{16}$$

$$P(A \cap B) = \frac{2}{16}$$

$$P(A \cup B) = \frac{10}{16}$$



Note that

$$P(A \cup B) \neq P(A) + P(B)$$

because $A \cap B \neq \emptyset$.

could use the formula

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(\bar{A}) = \frac{8}{16}$$

$$P(\bar{B}) = \frac{12}{16}$$

$$\textcircled{4} (a) S = \{1, 2, 3, 4\}$$

$$P(\{1\}) = \frac{2}{8} \quad P(\{2\}) = \frac{2}{8}$$

$$P(\{3\}) = \frac{3}{8} \quad P(\{4\}) = \frac{1}{8}$$

$$\textcircled{4} (b) A = \{1, 3\}$$

$$P(A) = P(\{1, 3\}) = P(\{1\}) + P(\{3\}) = \frac{2}{8} + \frac{3}{8} = \frac{5}{8}$$

$$\textcircled{4} (c) B = \{1, 2, 3\}$$

$$\begin{aligned} P(B) = P(\{1, 2, 3\}) &= P(\{1\}) + P(\{2\}) + P(\{3\}) \\ &= \frac{2}{8} + \frac{2}{8} + \frac{3}{8} = \frac{7}{8} \end{aligned}$$

Note: In this problem the elements of S are not equally weighted, so it is not true that $P(E) = \frac{|E|}{|S|}$.

⑤

$$S = \{(1,1), (1,2), (1,3), (1,4), (2,1), (2,2), (2,3), (2,4), (3,1), (3,2), (3,3), (3,4), (4,1), (4,2), (4,3), (4,4)\}$$

$P(\{(a,b)\}) = \frac{1}{16}$ for each (a,b) in S ,
that is the elements are equally weighted.
So, $P(E) = \frac{|E|}{|S|} = \frac{|E|}{16}$ for any event E .

⑤(a) Let A be the event that at least one of the dice shows a 2. Then,

$$A = \{(1,2), (2,1), (2,2), (2,3), (2,4), (3,2), (4,2)\}$$

$$P(A) = \frac{7}{16}$$

⑤(b) Let B be the event that the sum of the dice is a 4.

$$B = \{(1,3), (2,2), (3,1)\}$$

$$P(B) = \frac{3}{16}$$

⑤(c) Let C be the event that the sum of the dice is either 5 or 7.

$$C = \{(1,4), (4,1), (2,3), (3,2), (3,4), (4,3)\}$$

$$P(C) = \frac{6}{16}$$

⑥ The sample space is

$$S = \{(g, r) \mid g, r = 1, 2, 3, 4, 5, 6, 7, 8\}$$
$$= \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8),$$
$$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6), (2,7), (2,8),$$
$$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6), (3,7), (3,8),$$
$$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6), (4,7), (4,8),$$
$$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6), (5,7), (5,8),$$
$$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6), (6,7), (6,8),$$
$$(7,1), (7,2), (7,3), (7,4), (7,5), (7,6), (7,7), (7,8),$$
$$(8,1), (8,2), (8,3), (8,4), (8,5), (8,6), (8,7), (8,8)\}$$

All the elements are equally weighted so
if E is an event then $P(E) = \frac{|E|}{|S|} = \frac{|E|}{64}$



Let A be the event that the red die has a larger value than the green die. Then,

$$A = \{ (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), \\ (2,3), (2,4), (2,5), (2,6), (2,7), (2,8), \\ (3,4), (3,5), (3,6), (3,7), (3,8), \\ (4,5), (4,6), (4,7), (4,8), \\ (5,6), (5,7), (5,8), \\ (6,7), (6,8), \\ (7,8) \}$$

$$\text{Thus, } P(A) = \frac{|A|}{|S|} = \frac{28}{64}$$

$$\textcircled{7} \quad S = \{W, R, G\}$$

$$P(\{W\}) = \frac{1}{3} \quad P(\{R\}) = \frac{1}{3} \quad P(\{G\}) = \frac{1}{3}$$

$$\textcircled{8} \quad S = \{ \{W, R\}, \{W, G\}, \{R, G\} \}$$

There are 3 elements of S .

We use sets $\{W, R\}$ for example instead of (W, R) because we don't care about the order. And $(W, R) \neq (R, W)$ but $\{W, R\} = \{R, W\}$.

$\{W, R\}$ represents choosing 1 white and 1 red ball.

$\{W, G\}$ represents choosing 1 white and 1 green ball.

$\{R, G\}$ represents choosing 1 red and 1 green ball.

$$P(\{W, R\}) = \frac{1}{3} \quad P(\{W, G\}) = \frac{1}{3} \quad P(\{R, G\}) = \frac{1}{3}$$

9 Define

2 on first roll

2 on 2nd roll

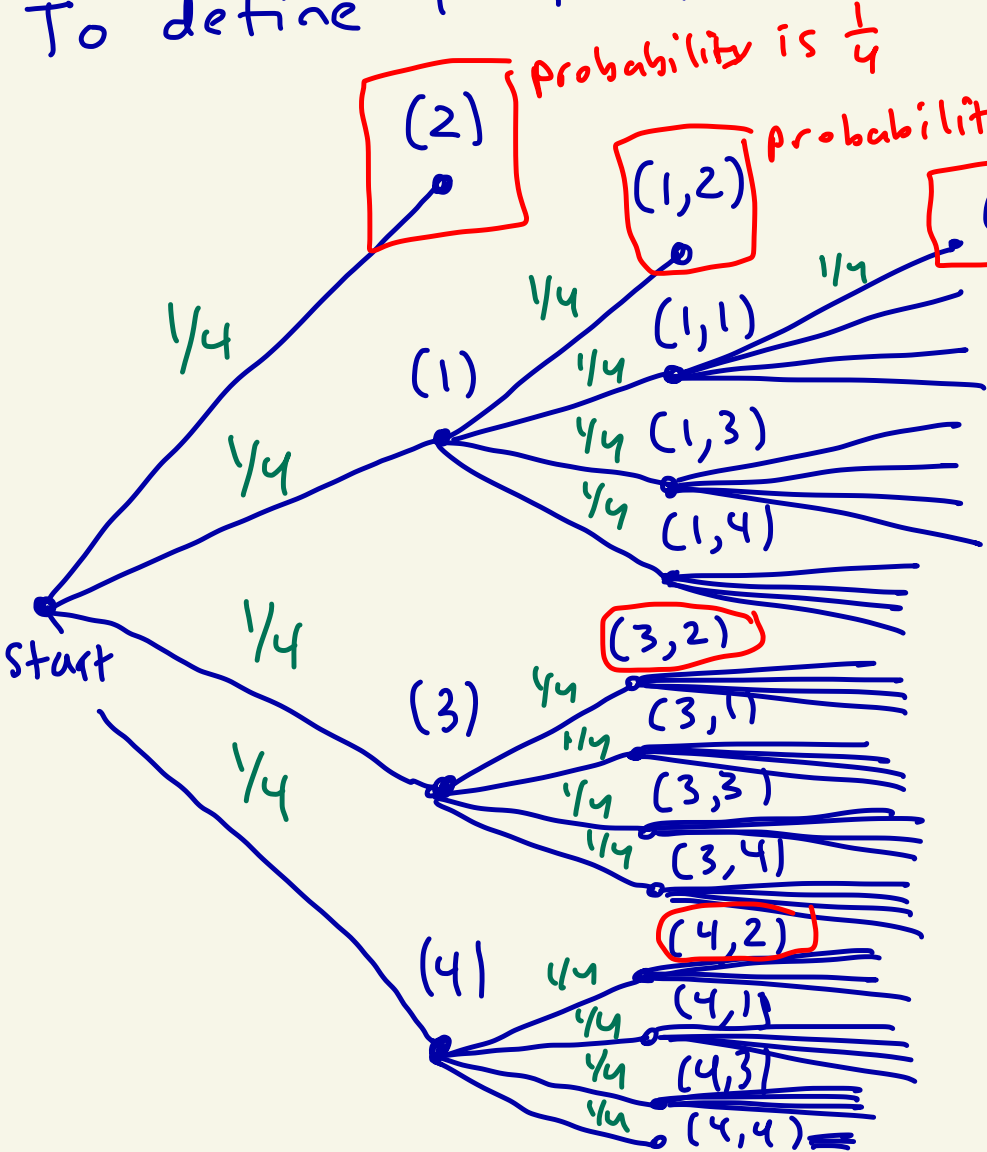
$$S = \{ (2), (1,2), (3,2), (4,2), (1,1,2), (1,3,2), (1,4,2), (3,1,2), (3,3,2), (3,4,2), (4,1,2), (4,3,2), (4,4,2), \dots \}$$

2 on 3rd roll

2 on 4th, 5th, 6th, ... roll

$$= \{ (2) \} \cup \{ (a_1, a_2, \dots, a_n, 2) \mid a_1, a_2, \dots, a_n = 1, 3, 4 \text{ and } n \geq 1 \}$$

To define P think of the experiment this way



Tree keeps going forever except stops at each red spot with probabilities in green

Define

$$P(\{2\}) = \frac{1}{4}$$

$$P(\{a_1, 2\}) = \left(\frac{1}{4}\right)^2 \text{ where } a_1 = 1, 3, 4$$

$$P(\{a_1, a_2, 2\}) = \left(\frac{1}{4}\right)^3 \text{ where } a_1, a_2 = 1, 3, 4$$

and in general

$$P(\{a_1, a_2, \dots, a_n, 2\}) = \left(\frac{1}{4}\right)^{n+1}$$

For example, $P(\{1, 2\}) = \left(\frac{1}{4}\right)^2$ and

$$P(\{1, 3, 2\}) = \left(\frac{1}{4}\right)^3.$$

Let Ω be all subsets of S .

If $E \in \Omega$, define $P(E) = \sum_{\omega \in E} P(\{\omega\})$.

We see that $0 \leq P(\{\omega\}) \leq 1$ for any $\omega \in S$. Now let's show

$$P(S) = 1.$$

We have that

$$P(S) = \sum_{w \in S} P(\{w\})$$

$$= P(\{(2)\})$$

$$+ P(\{(1,2)\}) + P(\{(3,2)\}) + P(\{(4,2)\})$$

$$+ P(\{(1,1,2)\}) + P(\{(1,3,2)\}) + P(\{(1,4,2)\})$$

$$+ P(\{(3,1,2)\}) + P(\{(3,3,2)\}) + P(\{(3,4,2)\})$$

$$+ P(\{(4,1,2)\}) + P(\{(4,3,2)\}) + P(\{(4,4,2)\})$$

+ ...

$$= \frac{1}{4} + 3 \cdot \left(\frac{1}{4}\right)^2 + 3^2 \cdot \left(\frac{1}{4}\right)^3$$

$$+ 3^3 \cdot \left(\frac{1}{4}\right)^4 + 3^4 \cdot \left(\frac{1}{4}\right)^5 + \dots$$

$$= \frac{1}{4} \left[1 + \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \dots \right]$$

Calculus

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

if $|x| < 1$

$$= \frac{1}{4} \left[\frac{1}{1 - \frac{3}{4}} \right]$$

$$= \frac{1}{4} [4] = 1.$$

Thus, $P(S) = 1$.

Since Ω is all subsets of S
we know that Ω satisfies axioms
①, ②, ③ of the def of probability
space.

We just verified ④, ⑤ axioms
for P .

Axiom ⑥ is true since we defined

$$P(E) = \sum_{w \in E} P(\{w\}).$$

(10) See the solution for #9 before reading this. We do the same idea, but I put more details in the solution for 9.

Define

$$S = \{ (H), (T, H), (T, T, H), \\ (T, T, T, H), (T, T, T, T, H), \dots \}$$

Let Ω be all subsets of S .



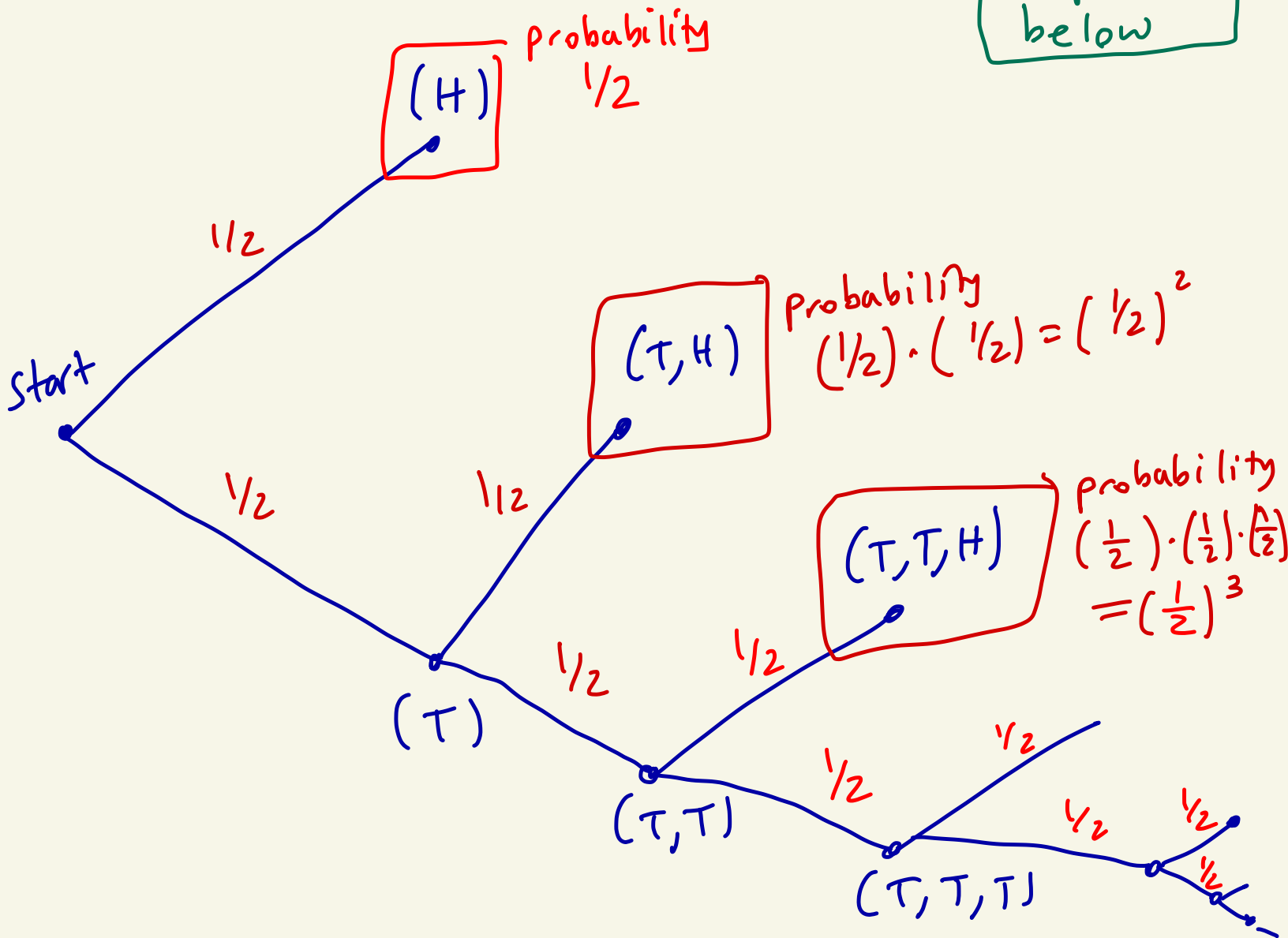
Define

$$P(\underbrace{\{T, T, \dots, T, H\}}_{k \text{ tails}}) = \underbrace{\left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)}_{\text{see picture}} = \left(\frac{1}{2}\right)^{k+1}$$

Define

$$P(\underbrace{\{T, T, \dots, T, H\}}_{k \text{ tails}}) = \underbrace{\left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)}_{\text{see picture}} = \left(\frac{1}{2}\right)$$

see picture below



Then for any event E define

$$P(E) = \sum_{\omega \in E} P(\{\omega\})$$

By how we defined Ω , Ω will satisfy axioms ①, ②, ③ of a probability space.

By how we defined P , P will satisfy axioms ④, ⑥ of a probability space.

Thus, we just need to verify axiom ⑤.

$$P(S) = \sum_{\omega \in S} P(\{\omega\})$$

$$= P(\{H\}) + P(\{T, H\}) \\ + P(\{T, T, H\}) + P(\{T, T, T, H\})$$

+ ...

$$= \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \dots$$

$$= \frac{1}{2} \left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots \right]$$

Calculus

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x}$$

if $|x| < \frac{1}{2}$

$$\Downarrow \frac{1}{2} \left[\frac{1}{1 - \frac{1}{2}} \right] = \frac{1}{2} [2] = 1.$$