Instructions:

• Do exactly two problems from Part A AND two problems from Part B. If you attempt more than two problems in either Part A or Part B, and do not clearly indicate which two are to count, only the first two problems will be counted towards your grade.

• No notes, books, calculators, internet or cell phones are allowed.

PART A: Do only TWO problems

1. (a) Let \( A = \begin{pmatrix} \alpha & \gamma \\ \gamma & \beta \end{pmatrix} \), where \( \alpha, \beta \) and \( \gamma \) are real numbers with \( \alpha > 0 \) and \( \beta > 0 \).
   
   i. Give the conditions on \( \alpha, \beta \) and \( \gamma \) under which \( A \) is strictly diagonally dominant. [2 points]
   
   ii. Find the eigenvalues (in terms of \( \alpha, \beta \) and \( \gamma \)) of the Jacobi iteration matrix when applied to solve the system \( Ax = b \) for some vector \( b \). [6 points]
   
   iii. Under what conditions on \( \alpha, \beta \) and \( \gamma \) will the Jacobi iteration converge? [3 points]

(b) Given the system of linear equations \( Bx = b \), where \( B \) is a strictly diagonally dominant \( n \times n \) matrix and \( b \) is an arbitrary \( n \)-vector. Prove that the Jacobi iteration matrix \( G_J \) for this system satisfies \( \|G_J\|_\infty < 1 \). [9 points]

(c) Determine whether the following statement is true or false: 
   
   *If a square matrix \( C \) is positive definite, then it is diagonally dominant.*
   
   If it is true, prove the statement. If it is false, give a counter example. [5 points]

2. (a) The Power Method and the QR Method are techniques for finding approximations to the eigenvalues of a square matrix \( A \).
   
   i. State the sufficient conditions for the convergence of the Power Method. [4 points]

   ii. Briefly describe the QR algorithm for finding the eigenvalues of \( A \). [5 points]

   iii. Let \( A = \begin{pmatrix} 1 & -3 \\ -2 & 2 \end{pmatrix} \). Perform one iteration of the QR Method to approximate the eigenvalues of \( A \) by letting \( A_0 = A \). [8 points]
iv. Give one advantage of the Power Method over the QR Method. [2 points]

(b) Find the 3 × 3 matrix $B$ that has eigenvalues $\lambda_1 = 4, \lambda_2 = 3, \lambda_3 = 2$ and the corresponding orthogonal eigenvectors $v_1 = (1/\sqrt{2}, 0, 1/\sqrt{2})^T, v_2 = (0, 1, 0)^T, v_3 = (-1/\sqrt{2}, 0, 1/\sqrt{2})^T$. [6 points]

3. (a) Let $A = \begin{pmatrix} 2 & 3 & 3 \\ 0 & 5 & 7 \\ 6 & 9 & 8 \end{pmatrix}$.

i. Use the Gaussian elimination with partial pivoting to write $A$ in the form $PA = LU$, where $P$ is a permutation matrix, $L$ is a unit lower triangular, and $U$ is an upper triangular matrix. [6 points]

ii. Use the result from part (i) to solve $Ax = b$ where $b = (1, 0, 5)^T$. [6 points]

(b) Let $B$ be an invertible $n \times n$ matrix. Show that if $B$ can be factored as $B = LU$, where $L$ is a unit lower triangular and $U$ is an upper-triangular, then this factorization is unique. [7 points]

(c) Compare (do not calculate) the flop-counts of Gaussian elimination with no pivoting, partial pivoting and complete pivoting for a general $n \times n$ system of linear equations. [6 points]
PART B: Do only TWO problems

1. Consider the elliptic partial differential equation (PDE)

\[ U_{xx} + U_{xy} + U_{yy} = f(x, y), \quad \text{for } (x, y) \in (0, 4/3) \times (0, 1), \]
\[ U(x, y) = 0, \quad \text{for } (x, y) \in \{0, 4/3\} \times [0, 1] \cup [0, 4/3] \times \{0, 1\}. \]

(a) On a regular grid with \( \Delta x = \Delta y = h \), find a consistent finite difference approximation to \( U_{xy} \) by applying central difference in the two coordinate directions. You need not prove that the approximation is consistent. [5 points]

(b) Use the standard 5-point stencil for \( U_{xx} + U_{yy} \) and the approximation in (a) to discretize the elliptic PDE on the grid below with the given ordering. First, write 6 linear equations by discretizing the PDE at each interior grid point. Then, express the scheme in the form of a linear system \( Au = f \). [9 points]

(c) Explain the process of finding the local truncation error of the finite difference scheme in (b). You need not find the expression of the local truncation error. [3 points]

(d) Use the scheme developed in (b) to explain the concepts of consistency and convergence. Does consistency imply convergence? Explain your answer. [8 points]

![Grid Diagram]

2. Given the parabolic PDE

\[ U_t - U_{xx} = 0 \quad \text{for } 0 < x < 1, \ t > 0, \]
\[ U(x, t) = 0 \quad \text{for } x \in \{0, 1\}, \ t > 0, \]
\[ U(x, 0) = x(1 - x) \quad \text{for } 0 \leq x \leq 1. \]

(a) Write a consistent finite difference approximation to the above PDE including the initial and boundary conditions. No need to show that the scheme is consistent. [5 points]
(b) Use your scheme in (a) to explain the following concepts:
   i. consistency [2 points]
   ii. stability [2 points]
   iii. convergence [2 points]

(c) Determine the stability of your scheme in (a). If your scheme is stable, determine whether it is conditionally stable or unconditionally stable. If your scheme is unstable, write a stable scheme. [8 points]

(d) Can a scheme be consistent but not stable? If so, give an example. If not, explain why not. [3 points]

(e) Can a consistent scheme be stable but not convergent? If so, give an example. If not, explain why not. [3 points]

3. (a) Consider the wave equation
   \[ U_{tt} = 4U_{xx}, \quad x \in \mathbb{R}, \quad t > 0, \quad a \in \mathbb{R}. \]

   i. Determine the two characteristic directions. Calculate and sketch the two characteristic curves passing through the point \((2, 2)\) in the \(xt\)-plane. [5 points]

   ii. Suppose the following two schemes are derived for approximating the equation

   \[
   \begin{align*}
   \text{Scheme (A):} & \quad u_{j+1}^{n+1} = au_j^n + bu_j^n + cu_{j-1}^n + du_{j-1}^{n-1}; \\
   \text{Scheme (B):} & \quad u_{j+1}^{n+1} = \alpha u_{j+1}^n + \beta u_j^n + \gamma u_{j-1}^{n+1} + \delta u_{j-1}^{n-1},
   \end{align*}
   \]

   where \(u_j^n\) is the approximation to \(U(x_j, t_n)\). Consider a fixed ratio \(k/h = 1\), where \(h = \Delta x\) and \(k = \Delta t\). Find the numerical domain of dependence of the grid point \((x_j, t_{n+1})\) for both schemes. What can be said about the convergence of schemes (A) and (B)? Justify your answer. [8 points]

(b) Consider the PDE
   \[ U_t = 2U_x, \quad 0 < x < 1, \quad t > 0 \]

   with periodic boundary condition, that is, \(U(0, t) = U(1, t)\).

   i. Consider the finite difference scheme
   \[
   \frac{u_j^{n+1} - u_j^n}{k} = 2\frac{u_{j+1}^n - u_{j-1}^n}{2h}
   \]

   for approximating the PDE. Here \(u_j^n\) approximates \(U(x_j, t_n)\). Show that the scheme is unstable when \(k\) is chosen proportional to \(h\). [7 points]

   ii. Write a stable scheme when \(k\) is chosen proportional to \(h\). No need to show that the scheme is stable. [5 points]