Complex Analysis Comprehensive Examination  

Akis*, Gutarts, Shaheen  

Do five of the following seven problems. If you attempt more than 5, the best 5 will be used.

1. Evaluate \[ \oint_{\gamma} \frac{1}{e^z - 1} \, dz \] where \( \gamma \) is the circle of radius 9 centered at 0.

2. a. Show that \( |e^{-2z}| < 1 \), if and only if, \( \text{Re} \, z > 0 \).

   b. Show that \[ \left| \oint_{\gamma} \frac{e^{-2z}}{z} \, dz \right| < \frac{3}{\sqrt{5}} \] where \( \gamma \) is the line segment from \( 2 + i \) to \( 5 + i \).

3. For each of the following real valued functions of two variables \( u(x, y) \), determine if there is a real valued function \( v(x, y) \) such that \( f(z) = f(x + iy) = u(x, y) + iv(x, y) \) is analytic. Either find \( v(x, y) \), or explain why such function does not exist.

   a. \( u(x, y) = \sin x - xy \)  
   b. \( u(x, y) = e^{-y} \sin x \)

4. Find the Laurent series expansion for \( f(z) = \frac{1}{z^2(1 - z)} \)

   valid on in each of the regions \( 0 < |z| < 1 \), \( 1 < |z| < \infty \), and find the residue of \( f(z) \) at \( z_0 = 0 \).

5. Suppose \( n \) is a positive integer. Show there are exactly \( n \) solutions counting multiplicity, to the equation \( e^z = 4z^n - 1 \) in the unit disk \( |z| < 1 \).

6. Consider the arcs \( C_1 \) defined by \( z_1(t) = e^{it} \) where \( 0 \leq t < \frac{3\pi}{2} \), and \( C_2 \) defined by \( z_2(t) = t + i(t - 1) \) where \( 0 \leq t \leq 1 \).
a. Draw the contour $C = C_1 + C_2$, and find its length.

b. Evaluate the integrals

$$\int_{C_1} \frac{dz}{z}, \quad \int_{C_2} \frac{dz}{z}, \quad \int_{C} \frac{dz}{z}.$$

7. Evaluate the following integrals by using residues:

a. $\int_0^\infty \frac{dx}{x^4 + 1}$

b. $\int_{-\infty}^{\infty} \frac{x \, dx}{(x^2 + 1)(x^2 + 2x + 2)}$