Spring 2021 # 1. Describe and sketch each of the following sets of complex numbers.

(a) \( \{ z \mid \overline{z} = \frac{1}{z} \} \)

(b) \( \{ e^z \mid z = x + iy \text{ and } 1 < x < 2 \text{ and } \frac{3\pi}{4} < y \leq \frac{5\pi}{4} \} \)

(c) \( \{ z \mid |z - i| \leq \text{Im}(z) \} \)

Spring 2021 # 2. Compute the following integrals.

(a) \( \int_{\gamma} \frac{e^{z^2}}{z^3} \, dz \) where \( \gamma \) is the unit circle oriented counter-clockwise

(b) \( \int_{0}^{\infty} \frac{x^2}{1 + x^4} \, dx \)

Spring 2021 # 3. Let \( f(z) = \frac{\sin(z)}{(e^z - 1)^2} \)

(a) Classify the singularity of \( f \) at \( z_0 = 0 \). That is, is it a removable singularity, a pole of order \( m \), or an essential singularity?

(b) Compute the integral \( \int_{\gamma} f(z) \, dz \) where \( \gamma \) is the unit circle oriented counter-clockwise
Spring 2021 # 4. Let \( p(z) = z^4 + 3z^3 + 6 \).

(a) Show that \( p(z) \) has three zeros (counting multiplicity) in the set \( \{ z \mid |z| < 2 \} \)

(b) Show that \( p(z) \) has one zero (counting multiplicity) in the set \( \{ z \mid 2 \leq |z| < 4 \} \)

Spring 2021 # 5. Prove that a sequence of complex numbers \( \{ z_n \} \) converges if and only if \( \{ z_n \} \) is Cauchy.

Note: You may use the fact that \( \mathbb{R} \) is complete.

Spring 2021 # 6. We say a function \( f: \mathbb{R} \to \mathbb{R} \) preserves orientation if \( f(x_1) < f(x_2) \) whenever \( x_1 < x_2 \), and reverses orientation if \( f(x_1) > f(x_2) \) whenever \( x_1 < x_2 \).

If possible, find an entire function \( g: \mathbb{C} \to \mathbb{C} \) such that

\[
\text{Im} \left[ g(x + i0) \right] = 0 = \text{Re} \left[ g(0 + iy) \right],
\]

and \( f_1: \mathbb{R} \to \mathbb{R} \) defined by \( f_1(x) = g(x + i0) \) preserves orientation, while \( f_2: \mathbb{R} \to \mathbb{R} \) defined by \( f_2(y) = g(0 + iy) \) reverses orientation. If not possible, prove that no such function \( g \) exists.

Spring 2021 # 7. If possible, find an entire function \( g: \mathbb{C} \to \mathbb{C} \) such that

\[
g'(z) = \begin{cases} 
z & \text{if } |z| < 1 \\
2z & \text{if } |z| > 2
\end{cases}
\]

If not possible, prove that no such function \( g \) exists.