THE SEPARATION OF VARIABLES METHOD FOR SECOND ORDER LINEAR PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

The Separation of Variables Method for Second Order

Linear Partial Differential Equations

By

Jorge Dimas Granados Del Cid

This thesis provides an overview of various partial differential equations, including their applications, classifications, and methods of solving them. We show the reduction (change of variables process) of an elliptic equation to the Laplace equation (with lower order terms), as well as other cases. We derive the solutions of some partial differential equations of 2nd order using the method of separation of variables.

The derivation includes various boundary conditions: Dirichlet, Neumann, mixed, periodic and Robin. A discussion of the eigenvalues related to various boundary conditions is provided. A discussion of Fourier series, as they apply to computing the coefficients of the series solutions, is included. The thesis concludes with a presentation of open problems related to the topic.

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CHAPTER 1

Introduction

A partial differential equation (PDE) is an equality composed of mathematical entities that include an unidentified, multivariable function and its partial derivatives.

Definition 1.1. *A partial derivative is the derivative with respect to one variable of a function of several variables, with the remaining variables treated as constants.*

For instance, the partial derivative of u with respect to x is

$$\lim_{h\to 0} \frac{u(x+h, y, z, \ldots) - u(x, y, z, \ldots)}{h}$$

These partial derivatives of u, with respect to independent variables such as x, y, $\dot{t}\ldots$, are written as

$$u_x, \quad , u_{xx}, \quad , u_{xy}, \quad , u_{xxx} \cdots \tag{1.0.1}$$

or

$$\frac{\partial}{\partial x}u, \quad \frac{\partial^2}{\partial x^2}u, \quad \frac{\partial^2}{\partial y \partial x}u, \quad \frac{\partial^3}{\partial x^3}u, \ldots$$

we write

$$u_{xy} \; = \; (u_x)_y \; = \; \frac{\partial^2}{\partial y \partial x}u,$$

to indicate that the partial with respect to x is taken first.

If we have a function of one variable, say x, then the only partial derivative of $f(x)$, $\dfrac{\partial f}{\partial x}$ is just the derivative $f'(x)$ and equations involving functions of one variable and their derivatives are called *ordinary differential equations* (ODEs).

Some example of PDEs are

$$u_x + tu_{tt} = t^2 \tag{1.0.2a}$$

$$u_t - k^2 u_{xx} = \cos t \tag{1.0.2b}$$

$$u_{tt} - c^2 u_{xx} + f(x,t) = 0 \tag{1.0.2c}$$

$$u_t + u u_x + u_{xxx} = 0 \tag{1.0.2d}$$

$$u_{xx}(x,y) + u_{yy}(x,y) = 0, \tag{1.0.2e}$$

$$u_{tt}(x,t) = u_{xx}(x,t) - u^3(x,t), \tag{1.0.2f}$$

$$u_t(x,t) + \left(x^2 + t^2\right) u_x(x,t) = 0, \tag{1.0.2g}$$

Definition 1.2. *The order of an ODE, or a PDE equation is the maximal number of derivatives (or partial derivatives, respectively) taken with respect to the independent variable(s).*

Equation (1.0.2a), is of second order because u has been differentiated twice with respect to t; equation (1.0.2d) has three x as subscripts, indicating a PDE of third order, and in (1.0.2g), the first u has been differentiated just once, and so has the second u; this is a PD equation of first order.

We now define what it means for a differential operator L to be linear.

Definition 1.3. *L is linear if for any fucntions u and v and constant c we have*

$$L(u+v) = Lu + Lv \qquad \& \qquad L(cu) = cLu$$

We show examples of *linearity* and *non-linearity* of the following PDEs in a slightly different way; we examine the factorization of the differential operator L and u in the following equations,

$$u_x + u_y = 0 \qquad \text{transport} \tag{1.0.3a}$$

$$u_x - yu_y = 0 \qquad \text{transport} \tag{1.0.3b}$$

$$u_x + uu_y = 0 \qquad \text{shock wave} \tag{1.0.3c}$$

We will write these equations in the form $Lu = u_x + u_y$. If we can factor out u completely from a differential operator L; that is, separate L from u with no u in L, then L is linear. We show this with equations (1.0.3a), (1.0.3b) and (1.0.3c)

$$Lu = u_x + u_y = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)u \tag{1.0.4a}$$

$$Lu = u_x - yu_y = \left(\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}\right)u \tag{1.0.4b}$$

$$Lu = u_x + uu_y = \left(\frac{\partial}{\partial x} + u\frac{\partial}{\partial y}\right)u \tag{1.0.4c}$$

The differential operator in (1.0.4a) is linear because there is no u in it; that is, no u in the differential operator $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)$; in (1.0.4b), the y in the differential operator makes no difference, so $u_x - yu_y$ is linear. And equation (1.0.4c) is not linear because there is a u in the differential operator after factoring.

Here are some operations that give nonlinear operators: u_x^2, u^3, $u_{xy}^4 \ldots$, $\sqrt{u_{xx}}$, $\ln(u)$, $\sin(u)$, $\cos(u), \ldots$, etc.

Using the definition of linearity we show $Lu = u_x + u_y$ is linear. For dependent functions v and u and c constant, we have

$$\begin{aligned} L(u+v) &= (u+v)_x + (u+v)_y \\ &= u_x + v_x + u_y + v_y \\ &= (u_x + u_y) + (v_x + v_y) \\ &= Lu + Lv, \end{aligned}$$

and

$$L(cu) = (cu)_x + (cu)_y = cu_x + cu_y = c(u_x + u_y) = cL(u)$$

this proves linearity of $Lu = u_x + u_y$.

Consider this nonlinear differential expression $Lu = u_x + u_y + 1$, then for dependent functions u and v,

$$L(u+v) = (u+v)_x + (u+v)_y + 1$$

$$= u_x + v_x + u_y + v_y + 1$$

but

$$Lu + Lv = u_x + u_y + 1 + v_x + v_y + 1$$

$$= u_x + v_x + u_y + v_y + 2$$

this means

$$L(u+v) \neq Lu + Lv,$$

hence, the operator L is nonlinear.

However, given

$$u_t - u_{xx} + 1 = 0$$

we can move the constant 1 to the right side

$$u_t - u_{xx} = -1,$$

and think of it as $Lu = -1$; the left side is linear (u can be factored from the differential operator L, or we can use the the definition of linearity above), so we have a linear equation.

Definition 1.4. *Suppose L is a linear operator, then we define $Lu = g$, where g is a function containing only independent variables such as $x, y, z, \ldots$ (u such as in u_x is not independent), then equation $Lu = g$ is called an inhomogeneous equation if*

$g \neq 0$ and $Lu = g$ is homogeneous if $g = 0$.

Examples of inhomogeneous (homogeneous) equations are

$$\left(\cos xy^2\right) u_x - y^2 u_y = \tan\left(x^2 + y^2\right) \tag{1.0.5a}$$

$$u_{tt} - c^2 u_{xx} + f\left(x, t\right) = 0, \qquad \text{where} \qquad f\left(x, t\right) \neq 0 \tag{1.0.5b}$$

Both equations, (1.0.5a and (1.0.5b) are inhomogeneous, while

$$u_{xx}(x, y) + u_{yy}(x, y) = 0$$

is a homogeneous equation.

CHAPTER 2

Reduction to Canonical Form

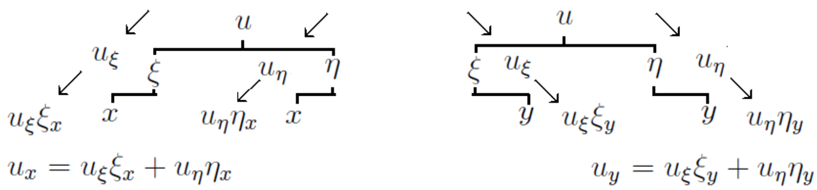

Figure 1: Chain rule

This chapter is dedicated to reducing three types of PDEs to their simplest possible forms, called *canonical form.* The most general case of second-order linear, partial differential equation (PDE) in two independent variables is given by

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{2.0.1}$$

where the coefficients A, B, and C are functions of x and y and do not vanish simultaneously...[1, p 57].

The second-order PDE (2.0.1) is classified by way of the discriminant

$$B^2 - 4AC$$

in the following definition

Definition 2.1. *The second order linear PDE (2.0.1) is called*

$$\textit{hyperbolic, if } \; B^2 - 4AC > 0$$

$$\textit{parabolic, if } \; B^2 - 4AC = 0$$

$$\textit{elliptic, if } \; B^2 - 4AC < 0.$$

Example 2.2.

$$u_{xx}(x,y) + u_{yy}(x,y) = 0 \tag{2.0.2}$$

$$bu_{xx}(x,y) - u_{yy}(x,y) = 0 \tag{2.0.3}$$

$$u_t - \gamma u_{xx} = 0 \tag{2.0.4}$$

Equation (2.0.2) has coefficients $A = 1$, $B = 0$ and $C = 1$; this gives $B^2 - 4AC < 0$, therefore, it is of the *elliptic* PDE type; while (2.0.3) has the discriminant of the form $B^2 - 4AC > 0$, connoting a *hyperbolic* PDE type; and (2.0.4) is a *parabolic* type, because $B = 0$, $C = 0$, and the constant $\gamma > 0$, so the discriminant $B^2 - 4AC = 0$.

The second-order PDE, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$, may be of one type at a set of points, and another type at some other points; this may happen if the coefficients contain independent variables such as x, $y, \ldots$ [2, p.2].

Theorem 2.3. *By a linear transformation of the independent variables, the equation*

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{2.0.1}$$

can be reduced (transformed) to one of the three (canonical) forms:

Hyperbolic case: *if* $B^2 - 4AC > 0$, *it is reducible to*

$$u_{\xi\xi} - u_{\eta\eta} + \cdots = 0,$$

Parabolic case: *if* $B^2 - 4AC = 0$, *it is reducible to*

$$u_{\xi\xi} + \cdots = 0,$$

Elliptic case: *if* $B^2 - 4AC < 0$, *it is reducible to*

$$u_{\xi\xi} + u_{\eta\eta} + \cdots = 0.$$

The dots represent the terms involving u and its first partial derivatives u_x and u_y

only. We will use the change of variables

$$\xi = \xi(x, y),\ \eta = \eta(x, y)$$

and the chain rule to transform the general equation $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$ into one of the *canonical* forms above.

The reason we refer to the equations in Theorem (2.3):

$$u_{\xi\xi} - u_{\eta\eta} + \cdots = 0,$$

$$u_{\xi\xi} + \cdots = 0,$$

$$u_{\xi\xi} + u_{\eta\eta} + \cdots = 0$$

> as *canonical forms* is that they correspond to particularly simple choices of the coefficients of the second partial derivatives of u. . . .
> [1, p. 58].

2.1 Chain rule with respect to change of variables

Reduction of $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$ to a *canonical form* starts by stating the partial derivatives of u with respect to x and y in terms of partials of ξ and η. v will be used on the right side when applying the chain rule to $u(x, y) = v(\xi(x, y), \eta(x, y))$ to compute the second-order functions $u_{xx},\ u_{xy},$ and u_{yy}:

$$\begin{aligned}
u_x &= v_\xi \xi_x + v_\eta \eta_x \\
(u_x)_x &= (v_\xi \xi_x)_x + (v_\eta \eta_x)_x \\
u_{xx} &= v_{\xi\xi}\xi_x^2 + v_\xi \xi_{xx} + v_{\xi\eta}\xi_x\eta_x + v_{\eta\xi}\eta_x\xi_x + v_{\eta\eta}\eta_x^2 + v_\eta \eta_{xx}
\end{aligned}$$

$$
\begin{aligned}
u_y &= v_\xi \xi_y + v_\eta \eta_y \\
(u_y)_y &= (v_\xi \xi_y)_y + (v_\eta \eta_y)_y \\
u_{yy} &= v_{\xi\xi}\xi_y^2 + v_\xi \xi_{yy} + v_{\xi\eta}\xi_y \eta_y + v_{\eta\xi}\eta_y \xi_y + v_{\eta\eta}\eta_y^2 + v_\eta \eta_{yy} \\
(u_x)_y &= (v_\xi \xi_x)_y + (v_\eta \eta_x)_y \\
u_{xy} &= v_{\xi\xi}\xi_x \xi_y + v_\xi \xi_{xy} + v_{\xi\eta}\xi_x \eta_y + v_{\eta\xi}\eta_x \xi_y + v_\eta \eta_{xy} + v_{\eta\eta}\eta_x \eta_y
\end{aligned}
$$

vanishing the first order terms $u_x,\ u_y$ (into the dots) we have

$$u_{xx} = v_{\xi\xi}\xi_x^2 + 2v_{\xi\eta}\xi_x \eta_x + v_{\eta\eta}\eta_x^2 + \cdots , \tag{2.1.1}$$

$$u_{xy} = v_{\xi\xi}\xi_x \xi_y + v_{\xi\eta}\xi_x \eta_y + v_{\xi\eta}\xi_y \eta_x + v_{\eta\eta}\eta_x \eta_y + \cdots , \tag{2.1.2}$$

$$u_{yy} = v_{\xi\xi}\xi_y^2 + 2v_{\xi\eta}\xi_y \eta_y + v_{\eta\eta}\eta_y^2 + \cdots , \tag{2.1.3}$$

Multiplying both sides of (2.1.1), (2.1.2), (2.1.3) by A, B, and C we have

$$Au_{xx} = v_{\xi\xi}A\xi_x^2 + 2v_{\xi\eta}A\xi_x \eta_x + v_{\eta\eta}A\eta_x^2 + \cdots , \tag{2.1.4}$$

$$Bu_{xy} = v_{\xi\xi}B\xi_x \xi_y + v_{\xi\eta}B\xi_x \eta_y + v_{\xi\eta}B\xi_y \eta_x + v_{\eta\eta}B\eta_x \eta_y + \cdots , \tag{2.1.5}$$

$$Cu_{yy} = v_{\xi\xi}C\xi_y^2 + 2v_{\xi\eta}C\xi_y \eta_y + v_{\eta\eta}C\eta_y^2 + \cdots , \tag{2.1.6}$$

We pick the new coefficients, $a,\ b,\ c,$ by gathering the multiplicative factors accompanying $v_{\xi\xi},\ v_{\xi\eta},\ v_{\eta\eta}$ in (2.1.4), (2.1.5), (2.1.6)

$$
\begin{aligned}
& v_{\xi\xi}A\xi_x^2 + v_{\xi\xi}B\xi_x \xi_y + v_{\xi\xi}C\xi_y^2 \\
= \; & \left(A\xi_x^2 + B\xi_x \xi_y + C\xi_y^2\right) v_{\xi\xi} \\
= \; & a v_{\xi\xi} \\
& 2v_{\xi\eta}A\xi_x \eta_x + Bv_{\xi\eta}(\xi_x \eta_y + \xi_y \eta_x) + 2v_{\xi\eta}C\xi_y \eta_y \\
= \; & (2A\xi_x \eta_x + B(\xi_x \eta_y + \xi_y \eta_x) + 2C\xi_y \eta_y)\, v_{\xi\eta}
\end{aligned}
$$

$$
\begin{aligned}
&= bv_{\xi\eta} \\
&\quad v_{\eta\eta}A\eta_x^2 + v_{\eta\eta}B\eta_x\eta_y + v_{\eta\eta}C\eta_y^2 \\
&= \left(A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2\right) v_{\eta\eta} \\
&= cv_{\eta\eta}
\end{aligned}
$$

consequently we have from the transformation $\xi = \xi(x,y), \quad \eta = \eta(x,y)$, and chain rule, new coefficients

$$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2, \tag{2.1.7}$$

$$b = 2A\xi_x\eta_x + (\xi_x\eta_y + \xi_y\eta_x)B + 2C\xi_y\eta_y, \tag{2.1.8}$$

$$c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2. \tag{2.1.9}$$

So equation $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$ becomes

$$av_{\xi\xi} + bv_{\xi\eta} + cv_{\eta\eta} + \cdots = 0 \tag{2.1.10}$$

with the dots representing the first order terms.

The Jacobian of the change of variables, $\xi = \xi(x,y)$ and $\eta = \eta(x,y)$, is

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \xi_x\eta_y - \xi_y\eta_x \neq 0$$

is not singular because

> Clearly we should confine our attention to locally one-to-one transformations whose Jacobians are different than zero....we conclude that the type of such an [general] equation, $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$, can not be altered by a real change of variables [1, p. 60].

The following formula multiplies the discriminant of the general PDE formula $Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$, by a positive number, implying that the transformation does not change its type.

$$(b)^2 - (4ac) = \left(B^2 - 4AC\right)(\xi_x\eta_y - \xi_y\eta_x)^2, \tag{2.1.11}$$

(where $(\xi_x\eta_y - \xi_y\eta_x)^2$ is a positve number) and leaves $B^2 - 4AC$ unchanged by the transformation of coordinates.

We prove (2.1.11) by plugging $a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2$,
$b = 2A\xi_x\eta_x + (\xi_x\eta_y + \xi_y\eta_x)B + 2C\xi_y\eta_y$ and $c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2$ into $b^2 - 4ac$

$$\begin{aligned}
&(b)^2 - (4ac) \\
&= \left(B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2A\xi_x\eta_x + 2C\xi_y\eta_y\right)^2 \\
&\quad -4\left(A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2\right)\left(A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2\right) \\
&= B^2\xi_x^2\eta_y^2 - 2B^2\xi_x\xi_y\eta_x\eta_y + B^2\xi_y^2\eta_x^2 - 4AC\xi_x^2\eta_y^2 + 8AC\xi_x\xi_y\eta_x\eta_y - 4AC\xi_y^2\eta_x^2,
\end{aligned}$$

and expanding the right side of (2.1.11) gives

$$\begin{aligned}
&\left(B^2 - 4AC\right)(\xi_x\eta_y - \xi_y\eta_x)^2 \\
&= B^2\xi_x^2\eta_y^2 - 2B^2\xi_x\xi_y\eta_x\eta_y + B^2\xi_y^2\eta_x^2 - 4AC\xi_x^2\eta_y^2 + 8AC\xi_x\xi_y\eta_x\eta_y - 4AC\xi_y^2\eta_x^2.
\end{aligned}$$

The two sides of (2.1.11) are the same. □

2.2 Hyperbolic reduction

We begin the reduction of $av_{\xi\xi} + bv_{\xi\eta} + cv_{\eta\eta} \cdots = 0$ to an equation of one of the *canonical forms* by taking coefficients a and c as polynomials and then complete the square to find the roots, this will make coefficients a and c zero and the reduction will result in a hyperbolic type PDE. We proceed in the fallowing manner; take both

$$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0, \qquad c = A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 = 0,$$

and divide each by ξ_y^2 and η_y^2 respectively, resulting in

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0, \qquad A\left(\frac{\eta_x}{\eta_y}\right)^2 + B\left(\frac{\eta_x}{\eta_y}\right) + C = 0$$

We find the roots completing the square

$$\begin{aligned}
A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C &= 0\\
\left(\frac{\xi_x}{\xi_y}\right)^2 + \frac{B}{A}\left(\frac{\xi_x}{\xi_y}\right) + \frac{C}{A} &= 0\\
\left(\frac{\xi_x}{\xi_y}\right)^2 + \frac{B}{A}\left(\frac{\xi_x}{\xi_y}\right) + \left(\frac{B}{2A}\right)^2 &= -\frac{C}{A} + \left(\frac{B}{2A}\right)^2\\
\left(\frac{\xi_x}{\xi_y} + \frac{B}{2A}\right)^2 &= \pm\left(\frac{B}{2A}\right)^2 - \frac{C}{A}\\
\frac{\xi_x}{\xi_y} + \frac{B}{2A} &= \pm\sqrt{\left(\frac{B}{2A}\right)^2 - \frac{C}{A}}\\
\frac{\xi_x}{\xi_y} + \frac{B}{2A} &= \pm\sqrt{\frac{B^2}{4A^2} - \frac{C}{A}}\\
\frac{\xi_x}{\xi_y} + \frac{B}{2A} &= \pm\sqrt{\frac{4A^2}{4A^2}\left(\frac{B^2}{4A^2} - \frac{C}{A}\right)}\\
\frac{\xi_x}{\xi_y} + \frac{B}{2A} &= \pm\sqrt{\frac{1}{4A^2}\left(B^2 - \frac{4A^2C}{A}\right)}\\
\frac{\xi_x}{\xi_y} + \frac{B}{2A} &= \pm\frac{1}{2A}\sqrt{\left(B^2 - \frac{4A^2C}{A}\right)}\\
\frac{\xi_x}{\xi_y} + \frac{B}{2A} &= \pm\frac{1}{2A}\sqrt{(B^2 - 4AC)}\\
\frac{\xi_x}{\xi_y} &= -\frac{B}{2A} \pm \frac{1}{2A}\sqrt{(B^2 - 4AC)}
\end{aligned}$$

we have the two roots $$\frac{\xi_x}{\xi_y} = \frac{-B \pm \sqrt{(B^2 - 4AC)}}{2A}$$

pick one root for $\frac{\xi_x}{\xi_y}$, and one root for $\frac{\eta_x}{\eta_y}$

$$\frac{\xi_x}{\xi_y} = \frac{-B - \sqrt{(B^2 - 4AC)}}{2A} \qquad \frac{\eta_x}{\eta_y} = \frac{-B + \sqrt{(B^2 - 4AC)}}{2A}.$$

The total derivative of ξ is

$$d\xi = \xi_x dx + \xi_y dy = 0,$$

along the coordinate line $\xi(x, y) =$ constant. The total derivative re-introduces original variables y and x by way of dy and dx by rearranging

$$d\xi = \xi_x dx + \xi_y dy = 0 \qquad \text{into} \qquad \frac{dy}{dx} = -\frac{\xi_x}{\xi_y}.$$

Similarly for $\eta(x, y) =$ constant we have

$$\frac{dy}{dx} = -\frac{\eta_x}{\eta_y}.$$

Replacing $\dfrac{\xi_x}{\xi_y}$ and $\dfrac{\eta_x}{\eta_y}$ above, we get

$$-\frac{dy}{dx} = \frac{-B - \sqrt{B^2 - 4AC}}{2A}, \qquad -\frac{dy}{dx} = \frac{-B + \sqrt{B^2 - 4AC}}{2A}. \tag{2.2.1}$$

Solving for dy in both equations we have

$$dy = \frac{B + \sqrt{B^2 - 4AC}}{2A} dx, \qquad dy = \frac{B - \sqrt{B^2 - 4AC}}{2A} dx$$

and integrate, making sure to add the constant of integration:

$$y = \frac{B + \sqrt{B^2 - 4AC}}{2A} x + c_1, \qquad y = \frac{B - \sqrt{B^2 - 4AC}}{2A} x + c_2.$$

We solve for c_1 and c_2

$$c_1 = y - \frac{B+\sqrt{B^2-4AC}}{2A}x, \qquad c_2 = y - \frac{B-\sqrt{B^2-4AC}}{2A}x$$

and the change of variables gives

$$\xi = y - \frac{B+\sqrt{B^2-4AC}}{2A}x, \qquad \eta = y - \frac{B-\sqrt{B^2-4AC}}{2A}x.$$

Taking partial derivatives of ξ and η with respect to x and y gives

$$\xi_x = -\frac{B+\sqrt{B^2-4AC}}{2A}, \qquad \eta_x = -\frac{B-\sqrt{B^2-4AC}}{2A},$$

$$\xi_y = 1, \qquad \eta_y = 1.$$

The Jacobian gives:

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = -\frac{1}{A}\sqrt{B^2-4AC} \neq 0.$$

Plug these partial derivatives, $\xi_x,\ \xi_y,\ \eta_x,\ \eta_y$, into the coefficients a, b, and c of

$$av_{\xi\xi} + bv_{\xi\eta} + cv_{\eta\eta}\cdots,$$

to get:

$$\begin{aligned}
a &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 \\
&= A\left(-\frac{1}{2A}\left(B+\sqrt{B^2-4AC}\right)\right)^2 \\
&\quad + B\left(-\frac{1}{2A}\left(B+\sqrt{B^2-4AC}\right)\right)(1) + C\,(1)^2 \\
&= \frac{1}{2A}B^2 - C + \frac{1}{2A}B\sqrt{B^2-4AC} + C \\
&\quad - \frac{1}{2A}B^2 - \frac{1}{2A}B\sqrt{B^2-4AC}
\end{aligned}$$

$$= 0$$

this gives $a = 0$. Now for c

$$\begin{aligned}
c &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2 \\
&= C + \frac{1}{4A}\left(B - \sqrt{B^2-4AC}\right)^2 \\
&\quad - \frac{1}{2A}B\left(B - \sqrt{B^2-4AC}\right) \\
&= C + \frac{1}{2A}B^2 - C - \frac{1}{2A}B\sqrt{B^2-4AC} \\
&\quad + \frac{1}{2A}B\sqrt{B^2-4AC} - \frac{1}{2A}B^2 \\
&= 0,
\end{aligned}$$

this gives $c = 0$. And last,

$$\begin{aligned}
b &= 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y \\
&= 2A\left(-\frac{B+\sqrt{B^2-4AC}}{2A}\right)\left(-\frac{B-\sqrt{B^2-4AC}}{2A}\right) \\
&\quad +B\left(-\frac{B+\sqrt{B^2-4AC}}{2A} - \frac{B-\sqrt{B^2-4AC}}{2A}\right) + 2C \\
&= 2C - \frac{1}{A}\left(B^2 - 2AC\right) \\
&= -\frac{1}{A}\left(B^2 - 4AC\right)
\end{aligned}$$

The left side of $av_{\xi\xi} + bv_{\xi\eta} + cv_{\eta\eta} \cdots = 0$ is reduced to

$$\begin{aligned}
&0 \cdot v_{\xi\xi} - \frac{1}{A}\left(B^2 - 4AC\right)v_{\xi\eta} + 0 \cdot v_{\eta\eta} + \cdots \\
&= 0 - \frac{1}{A}\left(B^2 - 4AC\right)v_{\xi\eta} + 0 + \cdots \\
&= -\frac{1}{A}\left(B^2 - 4AC\right)v_{\xi\eta} + \cdots \qquad \text{as needed}
\end{aligned}$$

Therefore, for $A \neq 0$,

$$v_{\xi\eta} + \cdots = 0 \tag{2.2.2}$$

is in (hyperbolic type) *canonical form.*

2.3 2^{nd} Hyperbolic Case

We need $av_{\xi\xi} + bv_{\xi\eta} + cv_{\eta\eta} + \cdots = 0$ reduced to the form

$$v_{\xi\xi} - v_{\eta\eta} + \cdots = 0.$$

$$\begin{aligned}
\text{Factor } \ 0 = v_{\xi\xi} - v_{\eta\eta} & \\
&= \left(\frac{\partial}{\partial\xi}\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\frac{\partial}{\partial\eta}\right) v \\
&= \left(\frac{\partial}{\partial\xi}\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\xi}\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\xi}\frac{\partial}{\partial\eta} - \frac{\partial}{\partial\eta}\frac{\partial}{\partial\eta}\right) v \\
&= \left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\right)\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right) v \ ;
\end{aligned}$$

this gives the gneneral solution for $v_{\xi\xi} - v_{\eta\eta} = 0$:

$$v(\xi, \eta) = f(\xi + \eta) + g(\xi - \eta).$$

Let $\alpha = \xi + \eta$, and $\beta = \xi - \eta$, and solve the system for ξ and η

$$\begin{aligned}
\xi + \eta &= \alpha \\
\xi - \eta &= \beta,
\end{aligned}$$

then we have

$$\xi = \frac{\beta + \alpha}{2}, \qquad \eta = \frac{\alpha - \beta}{2},$$

take partial derivatives of η and ξ with respect to α and β

$$\begin{aligned}\xi_\alpha &= \frac{1}{2} \quad \text{and} \quad \xi_\beta = \frac{1}{2}\\ \eta_\alpha &= \frac{1}{2} \quad \text{and} \quad \eta_\beta = -\frac{1}{2}\end{aligned}$$

The Jacobian gives

$$\frac{\partial(\xi,\eta)}{\partial(\alpha,\beta)} = \begin{vmatrix} \xi_\alpha & \xi_\beta \\ \eta_\alpha & \eta_\beta \end{vmatrix} = -\frac{1}{2} \neq 0.$$

State partial derivatives of u with respect to α and β, in terms of partial derivatives with respect to ξ and η of $u(\alpha,\beta) = v(\xi(\alpha,\eta),\eta(\alpha,\beta))$. By the chain rule

$$\begin{aligned}u_\alpha &= v_\xi \xi_\alpha + v_\eta \eta_\alpha\\ (u_\alpha)_\beta &= (v_\xi \xi_\alpha)_\beta + (v_\eta \eta_\alpha)_\beta\\ u_{\alpha\beta} &= v_{\xi\xi}\xi_\alpha\xi_\beta + v_\xi\xi_{\alpha\beta} + v_\xi \eta\xi_\alpha\eta_\beta + v_{\eta\xi}\eta_\alpha\xi_\beta + v_{\eta\eta}\eta_\alpha\eta_\beta + v_\eta\eta_{\alpha\beta}\\ u_{\alpha\beta} &= v_{\xi\xi}\xi_\alpha\xi_\beta + v_{\xi\eta}\xi_\alpha\eta_\beta + v_{\eta\xi}\eta_\alpha\xi_\beta + v_{\eta\eta}\eta_\alpha\eta_\beta + \cdots\end{aligned}$$

plugging partial derivatives: $\xi_\alpha = \frac{1}{2}$, $\xi_\beta = \frac{1}{2}$ and $\eta_\alpha = \frac{1}{2}, \eta_\beta = -\frac{1}{2}$, we have

$$\begin{aligned}u_{\alpha\beta} &= v_{\xi\xi}\frac{1}{2}\cdot\frac{1}{2} + \left[v_{\xi\eta}\frac{1}{2}\cdot\left(-\frac{1}{2}\right) + v_{\eta\xi}\frac{1}{2}\cdot\frac{1}{2}\right] + v_{\eta\eta}\frac{1}{2}\left(-\frac{1}{2}\right) + \cdots\\ &= \frac{1}{4}v_{\xi^2} + [0] - \frac{1}{4}v_{\eta^2} + \cdots\\ &= v_{\xi\xi} - v_{\eta\eta} + \cdots\\ &= 0.\end{aligned}$$

Then, $v_{\xi\xi} - v_{\eta\eta} + \cdots = 0$ is a linear, hyperbolic PDE in *canonical form.*

2.4 Parabolic Reduction

Now we reduce $av_{\xi\xi} + bv_{\xi\eta} + cv_{\eta\eta} \cdots = 0$ to a parabolic PDE in *canonical form*; this implies a and b or c and b must be zero, and $B^2 - 4AC = 0$. We set

$$a = A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2 = 0$$

and divide both sides of it by ξ_y^2 and get

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0\ . \qquad (2.4.1)$$

For $\xi(x, y) =$ constant, the total derivative is

$$d\xi = \xi_x dx + \xi_y dy = 0\ ,$$

and gives

$$-\xi_x dx = \xi_y dy \qquad \Longrightarrow \qquad -\frac{\xi_x}{\xi_y} = \frac{dy}{dx}$$

and modifies (2.4.1)

$$A\left(\frac{\xi_x}{\xi_y}\right)^2 + B\left(\frac{\xi_x}{\xi_y}\right) + C = 0 \qquad \text{becomes} \qquad A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0$$

and the quadratic formula gives one number since we must have $B^2 - 4AC = 0$; so, let

$$-\frac{dy}{dx} = -\frac{B \pm \sqrt{0}}{2A} = -\frac{B}{2A}.$$

So $\dfrac{dy}{dx} = \dfrac{B}{2A}$ gives

$$\begin{aligned} dy &= \frac{B}{2A}dx \\ y &= \frac{B}{2A}x + c_1 \end{aligned}$$

$$
\begin{aligned}
c_1 &= y - \frac{B}{2A}x \\
\Rightarrow \xi &= y - \frac{B}{2A}x
\end{aligned}
$$

and b gives

$$
\begin{aligned}
0 = b &= 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y \\
&= 2A\frac{\xi_x}{\xi_y}\eta_x + B\left(\frac{\xi_x}{\xi_y}\eta_y + \eta_x\right) + 2C\eta_y \\
&= 2A\left(-\frac{B}{2A}\right)\eta_x + B\left(\left(-\frac{B}{2A}\right)\eta_y + \eta_x\right) + 2C\eta_y \\
&= -B\eta_x - \frac{B^2}{2A}\eta_y + B\eta_x + 2C\eta_y \\
&= -\frac{B^2}{2A}\eta_y + 2C\eta_y \\
&= -B^2\eta_y + 4AC\eta_y \\
&= \left(B^2 - 4AC\right)\eta_y \ ,
\end{aligned}
$$

where $B^2 - 4AC = 0$, take η_y to be an arbitrary function of (x, y); η_y equals zero means that η is a constant, implying that it can be a function of x; so, letting $\eta = x$ gives the change of variables

$$
\xi = y - \frac{B}{2A} \qquad \text{and} \qquad \eta = x.
$$

with partial derivatives

$$
\begin{aligned}
\xi_x &= -\frac{B}{2A} \qquad \text{and} \qquad \xi_y = 1 \\
\eta_x &= 1 \qquad\quad\ \ \text{and} \qquad \eta_y = 0
\end{aligned}
$$

and the Jacobian is not zero:

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = -1 \neq 0. \tag{2.4.2}$$

The Jacobian is not singular, then the expression $a^2 - ab = (B^2 - AC)(\xi_x\eta_y - \xi_y\eta_x)^2$ holds and "shows that the sign of the discriminant $B^2 - AC$ remains invariant" [1, p. 60].

So, we reduce $av_{\xi\xi} + bv_{\xi\eta} + cv_{\eta\eta} \cdots = 0$ to a *canonical form* of the parabolic type; we take the partial derivatives of η and ξ with respect to x and y and plug them into each of the coefficients a, b, c as before,

$$\begin{aligned} a &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2, \\ b &= 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y, \\ c &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2, \end{aligned}$$

then we have

$$\begin{aligned} a &= A\left(-\frac{B}{2A}\right)^2 + B\left(-\frac{B}{2A}\right)(1) + C(1)^2 \\ &= C - \frac{1}{4A}B^2 \\ &= -\frac{1}{4A}\left(B^2 - 4AC\right) \\ &= 0 \\ &= -\frac{1}{4A}(0) \\ b &= 2A\left(-\frac{B}{2A}\right)(1) + B\left(\left(-\frac{B}{2A}\right)(0) + (1)(1)\right) \\ &\quad +2C(1)(0) \\ &= -B + B \end{aligned}$$

$$
\begin{aligned}
&= 0 \\
c &= A(1)^2 + B(1)(0) + C(0)^2 = A.
\end{aligned}
$$

So the equation $av_{\xi\xi} + bv_{\xi\eta} + cv_{\eta\eta} \cdots = 0$, reduces to $Av_{\eta\eta} \cdots = 0$,

$$\implies \quad v_{\eta\eta} \cdots = 0.$$

a parabolic PDE in *canonical form.*

2.5 Elliptic Canonical Reduction

The following is the reduction of $av_{\xi\xi} + bv_{\xi\eta} + cv_{\eta\eta} \cdots = 0$, to an elliptic type PDE *in canonical form* $\quad v_{\xi\xi} + v_{\eta\eta} \cdots = 0.$

The discriminant in this case should be $b^2 - 4ac < 0$; so we can set $b = 0$ and $c = a$, or $a - c = 0$.

So, with the coefficients:

$$
\begin{aligned}
a &= A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2 \\
b &= 2A\xi_x\eta_x + B(\xi_x\eta_y + \xi_y\eta_x) + 2C\xi_y\eta_y \\
c &= A\eta_x^2 + 2B\eta_x\eta_y + C\eta_y^2,
\end{aligned}
$$

we have that

$$
\begin{aligned}
a - c &= A\xi_x^2 + 2B\xi_x\xi_y + C\xi_y^2 - \left(A\eta_x^2 + 2B\eta_x\eta_y + C\eta_y^2\right) \\
&= A\left(\xi_x^2 - \eta_x^2\right) + B(\xi_x\xi_y - \eta_x\eta_y) + C\left(\xi_y^2 - \eta_y^2\right),
\end{aligned}
$$

this gives a "coupled" system; that is, both coordinates ξ and η, show up in both equations,

$$A\left(\xi_x^2-\eta_x^2\right)+B\left(\xi_x\xi_y-\eta_x\eta_y\right)+C\left(\xi_y^2-\eta_y^2\right)=0,$$

$$\text{and}\quad 2A\xi_x\eta_x+B\left(\xi_x\eta_y+\xi_y\eta_x\right)+2C\xi_y\eta_y=0.$$

In order to"separate" η and ξ, multiply

$$2A\xi_x\eta_x+B\left(\xi_x\eta_y+\xi_y\eta_x\right)+2C\xi_y\eta_y$$

by i and add the result to

$$A\left(\xi_x^2-\eta_x^2\right)+B\left(\xi_x\xi_y-\eta_x\eta_y\right)+C\left(\xi_y^2-\eta_y^2\right),$$

giving

$$A\left(\xi_x+i\eta_x\right)^2+B\left(\xi_x+i\eta_x\right)\left(\xi_y+i\eta_y\right)+C\left(\xi_y+i\eta_y\right)^2. \tag{2.5.1}$$

Dividing (2.5.1) by $\left(\xi_y+i\eta_y\right)^2$, results in

$$A\left(\frac{\xi_x+i\eta_x}{\xi_y+i\eta_y}\right)^2+B\left(\frac{\xi_x+i\eta_x}{\xi_y+i\eta_y}\right)+C=0,$$

a quadratic equation. Let $\left(\dfrac{\xi_x+i\eta_x}{\xi_y+i\eta_y}\right)=\phi$; dividing by A, we get

$$\phi^2+\frac{B}{A}\phi+\frac{C}{A}=0,$$

this gives the roots:

$$\begin{aligned}\phi^2+\frac{B}{A}\phi+\left(\frac{B}{2A}\right)^2&=-\frac{C}{A}+\left(\frac{B}{2A}\right)^2\\ \left(\phi+\frac{B}{2A}\right)^2&=\left(\frac{B}{2A}\right)^2-\frac{C}{A}\end{aligned}$$

$$\begin{aligned}
\phi + \frac{B}{2A} &= \pm\sqrt{\left(\frac{B}{2A}\right)^2 - \frac{C}{A}} \\
\phi + \frac{B}{2A} &= \pm\sqrt{\left(\frac{B}{2A}\right)^2 - \frac{C}{A}} \\
\phi + \frac{B}{2A} &= \pm\sqrt{\frac{B^2}{4A^2} - \frac{C}{A}} \\
\phi + \frac{B}{2A} &= \pm\sqrt{\frac{4A^2}{4A^2}\left(\frac{B^2}{4A^2} - \frac{C}{A}\right)} \\
\phi + \frac{B}{2A} &= \pm\sqrt{\frac{1}{4A^2}\left(\frac{B^2}{4A^2} - \frac{4A^2C}{A}\right)} \\
\phi + \frac{B}{2A} &= \pm\frac{1}{2A}\sqrt{\left(\frac{4A^2B^2}{4A^2} - \frac{4A^2C}{A}\right)} \\
\phi + \frac{B}{2A} &= \pm\frac{1}{2A}\sqrt{(B^2 - 4AC)} \\
\phi &= -\frac{B}{2A} \pm \frac{1}{2A}\sqrt{(B^2 - 4AC)} \\
\phi &= \frac{-B \pm \sqrt{(B^2 - 4AC)}}{2A}.
\end{aligned}$$

A PDE of an elliptic type implies that $(B^2 - 4AC) < 0,$ so we have that

$$\begin{aligned}
\phi &= \frac{-B \pm \sqrt{(B^2 - 4AC)}}{2A} \\
&= \frac{-B \pm \sqrt{-1\,(4AC - B^2)}}{2A} \\
&= \frac{-B \pm i\sqrt{(4AC - B^2)}}{2A},
\end{aligned}$$

we have complex roots. Let $\phi = \dfrac{\alpha_x}{\alpha_y}$ for one root, and $\phi = \dfrac{\beta_x}{\beta_y}$ for the other root; so,

$$\frac{\alpha_x}{\alpha_y} = \frac{-B + i\sqrt{(4AC - B^2)}}{2A}, \qquad \frac{\beta_x}{\beta_y} = \frac{-B - i\sqrt{(4AC - B^2)}}{2A},$$

these are complex conjugates. The total derivative replaces $\dfrac{\alpha_x}{\alpha_y}$ and $\dfrac{\beta_x}{\beta_y}$:

$$d\alpha = \alpha_x dx + \alpha_y dy = 0$$

$$\text{gives} \qquad \frac{dy}{dx} = -\frac{\alpha_x}{\alpha_y};$$

similarly, for $d\beta = \beta_x dx + \beta_y dy = 0$, we have

$$\frac{dy}{dx} = -\frac{\beta_x}{\beta_y},$$

then we have

$$dy = \frac{B - i\sqrt{(4AC - B^2)}}{2A}dx \; , \qquad dy = \frac{B + i\sqrt{(4AC - B^2)}}{2A}dx,$$

integrating both gives

$$y = \frac{B - i\sqrt{(4AC - B^2)}}{2A}x + c_1 \; , \qquad y = \frac{B + i\sqrt{(4AC - B^2)}}{2A}x + c_2,$$

and solving for c_1 snf c_2 we get,

$$c_1 = y - \frac{B - i\sqrt{(4AC - B^2)}}{2A}x \; , \qquad c_2 = y - \frac{B + i\sqrt{(4AC - B^2)}}{2A}x.$$

Let $c_1 = \alpha$, and $c_2 = \beta$

$$\alpha = y - \frac{B - i\sqrt{(4AC - B^2)}}{2A}x \; , \qquad \beta = y - \frac{B + i\sqrt{(4AC - B^2)}}{2A}x, \qquad (2.5.2)$$

to find what is α and β in terms of η and ξ we again do the factoring:

$$\begin{aligned}
v_{\xi\xi} + v_{\eta\eta} &= \left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\right) v \\
&= \left(\frac{\partial}{\partial\xi}\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\frac{\partial}{\partial\eta}\right) v \\
&= \left(\frac{\partial}{\partial\xi}\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\xi}\frac{\partial}{\partial\eta} - \frac{\partial}{\partial\xi}\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\eta}\frac{\partial}{\partial\eta}\right) v \\
&= \left[\frac{\partial}{\partial\xi}\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right) - \frac{\partial}{\partial\eta}\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right)\right] v
\end{aligned}$$

$$= \left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right)\left(\frac{\partial}{\partial\xi} - \frac{\partial}{\partial\eta}\right)v,$$

this gives the general solution,

$$u(\xi,\eta) = f(\xi+\eta) + g(\xi-\eta).$$

It follows that the (complex) characteristic coordinates are

$$\alpha = \xi + i\eta, \qquad \text{and} \qquad \beta = \xi - i\eta,$$

solving for η and ξ, we have

$$\xi = \frac{1}{2}\alpha + \frac{1}{2}\beta \qquad \text{and} \qquad \eta = \frac{1}{2}i\beta - \frac{1}{2}i\alpha.$$

The equations found above:

$$\alpha = y - \frac{B - i\sqrt{(4AC - B^2)}}{2A}x\,, \qquad \beta = y - \frac{B + i\sqrt{(4AC - B^2)}}{2A}x,$$

give $$\xi = y - \frac{1}{2A}Bx \quad \text{and} \quad \eta = \frac{1}{2A}x\sqrt{4AC - B^2}.$$

Taking partial derivatives of ξ and η we have

$$\begin{aligned} \eta_y &= 0, & \eta_x = \frac{1}{2A}\sqrt{AC - B^2}, \\ \xi_x &= -\frac{1}{2A}B, & \xi_y = 1. \end{aligned}$$

The Jacobian gives

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = -\frac{1}{2A}\sqrt{AC - B^2} \neq 0.$$

Now plug these numbers into

$$av_{\xi\xi} + bv_{\xi\eta} + cv_{\eta\eta} \cdots = 0.$$

We have $a = c$, and need b equal to zero,

$$
\begin{aligned}
b &= 2A\xi_x\eta_x + B\left(\xi_x\eta_y + \xi_y\eta_x\right) + 2C\xi_y\eta_y \\
&= 2A\left(-\frac{1}{2A}B\right)\left(\frac{1}{2A}\sqrt{AC-B^2}\right) + B\left(-\frac{1}{2A}B\right)(0) \\
&\quad +B\,(1)\left(\frac{1}{2A}\sqrt{AC-B^2}\right) + 2C\,(1)\,(0) \\
&= -\frac{1}{2A}B\sqrt{AC-B^2} + \frac{1}{2A}B\sqrt{AC-B^2} \\
&= 0
\end{aligned}
$$

and since we set $a = c$, we get

$$
\begin{aligned}
0 &= av_{\xi\xi} + 0\cdot v_{\xi\eta} + av_{\eta\eta}\cdots \\
&= a\left(v_{\xi\xi} + v_{\eta\eta}\right)\cdots\,, \\
\text{this implies that}\qquad 0 &= v_{\xi\xi} + v_{\eta\eta}\cdots
\end{aligned}
$$

is a PDE of an elliptic type in *canonical form*.

CHAPTER 3

Separation Of Variables

3.1 Examples of second-order PDEs and Boundary Conditions

Some well known examples of PDEs are

$$u_{tt} = c^2 u_{xx} \qquad \text{wave equation} \tag{3.1.1}$$

$$u_t = k u_{xx} \qquad \text{diffusion equation} \tag{3.1.2}$$

$$u_{xx} + u_{yy} = 0 \qquad \text{Laplace equation,} \tag{3.1.3}$$

where each represent a different type: the wave equation represents 'hyperbolic' equations; while the Laplace equation is an elliptic type of equation, and the heat equation's type is parabolic. Some conditions have to be given for the equations above

> Because PDEs typically have so many solutions,... we single out one solution by imposing auxiliary condition... (Strauss, p. 20)

The following are some of the types of boundary conditions.

Definition 3.1. *Three well-known boundary conditions are*

(a) *The Dirichlet boundary conditions*

$$u(0,t) = g(t) \quad \textit{and} \quad u(l,t) = h(t), \quad \textit{where} \quad 0 < x < l,$$

is used when u is given at the boundary.

(b) *The Neumann boundary conditions with the form*

$$u_x(0,t) = f_1(t) \quad \textit{and} \quad u_x(l,t) = f_2(t), \quad \textit{where} \quad 0 < x < l,$$

is used when $\dfrac{\partial u}{\partial n}$ *is given at the boundary.*

(c) *The Robin boundary conditions with the form*

$$\begin{cases} u_x - a_0 u = f_3(t) & at\ x = 0 \\ u_x + a_l u = f_4(t) & at\ x = l \end{cases}$$

is used when $\dfrac{\partial u}{\partial n}$ *and* u *is applied at the boundary.*

3.2 The characteristic polynomial

Solving a linear, homogeneous, second order ODEs using the characteristic polynomial method.

We have the equation

$$aX^{''} + bX^{'} + cX = 0 \tag{3.2.1}$$

where $X = X(x)$, with a, b and c constants. We look for a solution $X(x)$ in the form $X = e^{rx}$, then

$$\begin{aligned} X^{'} &= re^{rx}, \\ X^{''} &= r^2 e^{rx}, \end{aligned}$$

use this to insert into $aX^{''} + bX^{'} + cX = 0$, this gives

$$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0'$$

and factoring e^{rx} we have

$$e^{rx}\left(ar^2 + br + c\right) = 0,$$

which gives the characteristic polynomial

$$ar^2 + br + c = 0. \tag{3.2.2}$$

The roots of this polynomial are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{3.2.3}$$

These roots, (3.2.3), give solution cases for the differential equation

$$aX^{''} + bX^{'} + cX = 0 \quad \text{where} \quad X(x) = e^{rx}.$$

If $\quad r_1 \neq r_2 \quad$ are two real solution, then $\quad X(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$;

if $\quad r_1 = r_2 \quad$ the solution has the form $\quad X(x) = c_1 e^{r_1 x} + c_2 \boldsymbol{x} e^{r_1 x}$

if $\quad r_1 \neq r_2$ & $b^2 - 4ac < 0$ (these are complex roots) the solutions have the form

$$X(x) = c_1 e^{(\kappa + i\gamma)x} + c_2 e^{(\kappa - i\gamma)x},$$

where $\kappa = \dfrac{-b}{2a}$ and $\gamma = \dfrac{\sqrt{4ac - b^2}}{2a}$.

By Euler's formula:

$$e^{i\gamma} = \cos\gamma x + i \sin\gamma x,$$

we get the following:

$$\begin{aligned}
X(x) &= c_1 e^{(\kappa + i\gamma)x} + c_2 e^{(\kappa - i\gamma)x} \\
&= e^{\kappa x}\left[\left(c_1 e^{i\gamma x}\right) + \left(c_2 e^{-i\gamma x}\right)\right] \\
&= e^{\kappa x}\left[c_1 (\cos\gamma x + i\sin\gamma x) + c_2 (\cos\gamma x - i\sin\gamma x)\right] \\
&= e^{\kappa x}\left[(c_1\cos\gamma x + ic_1\sin\gamma x) + (c_2\cos\gamma x - ic_2\sin\gamma x)\right] \\
&= e^{\kappa x}\left[(c_1 + c_2)\cos\gamma x + (c_1 - c_2)\, i\sin\gamma x\right] \\
&= e^{\kappa x}\left[C_1\cos\gamma x + C_2\sin\gamma x\right]
\end{aligned}$$

where $c_1 + c_2 = C_1$, and $(c_1 + c_2)i = C_2$. [3, Section 3.4].

3.3 Dirichlet boundary conditions

We look at the wave equation with Dirichlet boundary conditions:

$$\begin{aligned} u_{tt} &= c^2 u_{xx} \quad \text{for} \quad 0 < x < l && (3.3.1) \\ u(0,t) &= 0 = u(l,t) && (3.3.2) \end{aligned}$$

and initial conditions

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x). \tag{3.3.3}$$

We look for a solution as a separable function of x and t, of the form

$$u(x,t) = X(x)T(t), \tag{3.3.4}$$

next is to have $XT'' = c^2 X''T$ be divided by c^2XT

$$\frac{X(x)T''(t)}{c^2XT} = \frac{c^2X''T}{c^2XT} = -\lambda,$$

where $\lambda > 0$ is a constant. So, we get

$$X'' + \lambda X = 0 \quad \text{and} \qquad T'' + c^2\lambda T = 0,$$

where for convenience we let $\beta^2 = \lambda$ and $\beta > 0$. Then, we have

$$X'' + \beta^2 X = 0, \quad \text{and} \qquad T'' + c^2\beta^2 T = 0.$$

And by the characteristic method $(ar^2 + br + c = 0)$ we obtain $a = 1$, $b = 0$ and $c = \beta^2$, this implies that since $r_1 \neq r_2$ & $0^2 - 4\beta^2 < 0$, the solution has the form

$$X(x) = A\cos(\beta x) + B\sin(\beta x),$$

where $\kappa = 0$ and $\gamma = \beta$ in $e^{\kappa x}\left[C_1 \cos \gamma x + C_2 \sin \gamma x\right].$

Similarly for $T'' + c^2\beta^2 T = 0$, with $\kappa = 0$ and $\gamma = c\beta$, we have

$$T\left(t\right) = C \cos\left(c\beta t\right) + D \sin\left(c\beta t\right).$$

Using Dirichlet condition $u(0,t) = 0$ for X on the left we have

$$\begin{aligned} 0 = X\left(0\right) &= A \cos\left(\beta 0\right), + B \sin\left(\beta 0\right) \\ &= A(1) + B(0) \\ \text{this gives } A &= 0. \end{aligned}$$

So $X\left(x\right) = A \cos\left(\beta x\right) + B \sin\left(\beta x\right)$ becomes $X(x) = B \sin(\beta x)$ and the right boundary condition, $u(l,t) = 0$, says

$$X(l) = B \sin(\beta l) = 0.$$

If $B = 0$, then $u(x,y) = X(x)T(t) = X(0)T(t) = 0$; this is trivial. Instead, we let $\beta l = n\pi$, where $n\pi$ are zeros (roots) of the sine function; so, $\beta = \dfrac{n\pi}{l}$, and

$$\beta_n^2 = \left(\frac{n\pi}{l}\right)^2 = \lambda_n, \qquad X_n(x) = \sin\left(\frac{\pi x}{l}\right) \qquad (n = 1, 2, 3, \ldots)$$

For T, there are no boundary conditions so we have that $u(x,t) = X(x)T(t)$ becomes

$$u_n(x,t) = \left(C_n \cos \frac{n\pi c t}{l} + D_n \sin \frac{n\pi c t}{l}\right) \sin \frac{n\pi x}{l}, \qquad (3.3.5)$$

n is a set of integers from one to infinity and C_n and D_n are constants. So, we have infinitely many solutions because for each n, we have a different u_n and we have infinitely many n values from 1 to ∞. By the method of superposition, we get a sum of solutions as another solution of $u_{tt} = c^2 u_{xx}$ for $0 < x < l$ and $u(0,t) = 0 = u(l,t)$:

$$u(x,t) = \sum_n \left(C_n \cos \frac{n\pi ct}{l} + D_n \sin \frac{n\pi ct}{l} \right) \sin \frac{n\pi x}{l}, \tag{3.3.6}$$

and (3.3.6) "solves equations (3.3.1), (3.3.2) and (3.3.3) provided that $\phi(x) = \sum A_n \sin \frac{n\pi x}{l}$ and $\psi(x) = \sum \frac{n\pi c}{l} B_n \sin \frac{n\pi x}{l}$" [4, p. 89].

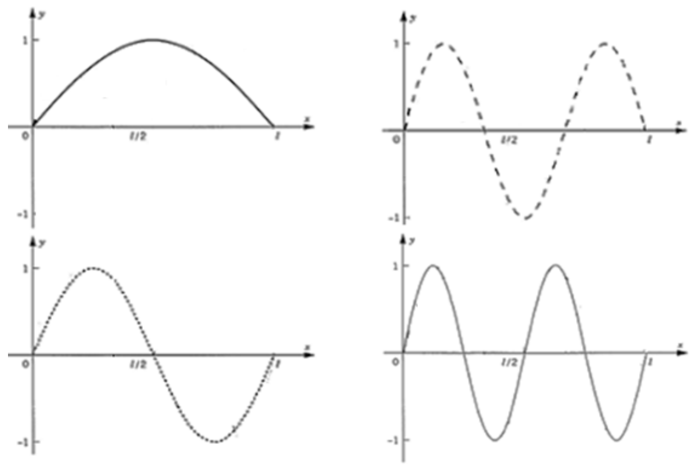

Figure 2: $\sin(n\pi x/l) \quad n = 1,2,3,4. \quad 0 < x < l.$

3.4 The Heat Equation with Dirichlet Boundaries

The diffusion problem with the Dirichlet boundary condition.

diffusion equation: $u_t = ku_{xx} \quad (0 < x < l,\ 0 < t < \infty)$ (3.4.1)

boundary condition: $u(0,t) = u(l,t) = 0$ (3.4.2)

initial condition: $u(x,0) = \phi(x).$ (3.4.3)

Continuing with the separation of variables method, we have

$$\frac{T'}{kT} = \frac{X''}{X} = -\lambda = \text{ constant},$$

and, as before, the boundary conditions give for $-X'' = \lambda X$

$$\lambda_n = \left(\frac{n\pi}{l}\right)^2, \quad X_n(x) = \sin\frac{n\pi x}{l} \quad (n = 1, 2, 3, \ldots). \tag{3.4.4}$$

For T (with $\dfrac{T'}{T} = \dfrac{d}{dt}\ln|T|$), we have

$$\begin{aligned} 0 = \frac{T'}{T} &= -\beta^2 k \\ \frac{d}{dt}\ln|T| &= -\beta^2 k \\ \ln|T_n| &= -\left(\frac{n\pi}{l}\right)^2 kt + c \\ T_n &= e^{-\left(\frac{n\pi}{l}\right)^2 kt + c} \\ &\equiv A_n e^{-\left(\frac{n\pi}{l}\right)^2 kt}. \end{aligned}$$

So, $u(x,t) = X(x)T(t)$ is now

$$u(x,t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \sin\frac{n\pi x}{l} \tag{3.4.5}$$

Note that T_n is an exponential function; also note that we have just one initial condition because the diffusion equation contains only one partial derivative with respect t. Imposing this initial condition $\phi(x)$, we get the following

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}. \tag{3.4.6}$$

Example 3.2. Consider waves in a resistant medium that satisfy the problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} - r u_t \quad for \ \ 0 < x < l, \\ \text{BC: } u &= 0 \quad \text{at both ends,} \\ \text{IC: } u(x,0) &= \phi(x), \quad u_t(x,0) = \psi(x), \end{aligned}$$

where r is a constant and $0 < r < \dfrac{2\pi c}{l}$. Write down the series expansion of the solution.

Solution: look for

$$u(x,t) = X(x)\,T(t),$$

write

$$u_t = X(x)\,T'(t), \qquad u_{tt} = X(x)\,T''(t) \qquad \text{and} \qquad u_{xx} = X''(x)\,T(t).$$

Writing $u_{tt} - c^2 u_{xx} + r u_t = 0$, we have

$$X(x)\,T''(t) - c^2 X(x)\,T''(t) + rX(x)T'(t) = 0,$$

and dividing by $c^2 X(x)\,T(t)$ gives

$$\frac{X(x)\,T''(t)}{c^2 X(x)\,T(t)} - \frac{X''(x)\,T(x)}{c^2 X(x)\,T(t)} + r\frac{X(x)\,T'(t)}{c^2 X(x)\,T(t)} = \frac{0}{c^2 X(x)\,T(t)},$$

then

$$\frac{T''(t)}{c^2 T(t)} - \frac{X''(x)}{X(x)} + r\frac{T'(t)}{c^2 T(t)} = 0,$$

insert $-\lambda$ and separate

$$\frac{T''(t)}{c^2 T(t)} + r\frac{T'(t)}{c^2 T(t)} = -\lambda \quad \text{and} \quad \frac{X''(x)}{X(x)} = -\lambda,$$

and solving X as before we have:

$$\left.\begin{array}{c} X''(x) + \lambda X(x) = 0 \\ u = 0 \text{ at both ends} \end{array}\right\} \Longrightarrow X_n = \sin\frac{n\pi x}{l} \quad \text{and} \quad \lambda_n = \left(\frac{n\pi}{l}\right)^2.$$

For T we have

$$\frac{1}{c^2}T''(t) + \frac{r}{c^2}T'(t) + \lambda T(t) = 0,$$

and multiplying c^2 on both sides gives:

$$T''(t) + rT'(t) + \lambda c^2 T(t) = 0.$$

We use the characteristic equation $\mu^2 + r\mu + \lambda c^2 = 0$, to find the roots for T; by the quadratic formula we get the roots:

$$\mu_n = \frac{-r \pm \sqrt{r^2 - 4c^2\lambda_n}}{2} = \frac{-r \pm \sqrt{r^2 - 4\left(\frac{n\pi}{l}\right)^2 c^2}}{2}$$

Given that $0 < r < \dfrac{2\pi c}{l}$, we have

$$r < \frac{2\pi c}{l} \Longrightarrow r^2 < \left(\frac{2\pi c}{l}\right)^2 = \frac{4\pi^2 c^2}{l^2} \le \frac{4\pi^2 c^2 n^2}{l^2}, \quad n = 1, 2, 3, \ldots,$$

so

$$\mu_n = -\frac{r \pm i\sqrt{4\left(\frac{n\pi}{l}\right)^2 c^2 - r^2}}{2} = -\frac{r}{2} \pm i\sqrt{\left(\frac{n\pi}{l}\right)^2 c^2 - \frac{r^2}{4}}.$$

Then,

$$T_n(t) = e^{\left(-\frac{r}{2} \pm i w_n\right)t}, \quad \text{where } w_n = \sqrt{4\left(\frac{n\pi}{l}\right)^2 c^2 - \frac{r^2}{4}}$$

and

$$\begin{aligned} T_n(t) &= e^{\left(-\frac{r}{2} \pm i w_n\right)t} \\ &= e^{-\frac{r}{2}t} e^{\pm i w_n t} \\ &= e^{-\frac{r}{2}t} \left(\cos w_n t \pm i \sin w_n t\right). \end{aligned}$$

Hence,

$$u_n(x,t) = e^{-\frac{r}{2}t} \sum_{n=1}^{\infty} \left(\cos w_n t + i \sin w_n t\right) \sin \frac{n\pi x}{l}, \qquad \text{where} \qquad w_n = \sqrt{4\left(\frac{n\pi}{l}\right)^2 c^2 - \frac{r^2}{4}}.$$

Note: $r > 0$, so $u(x,t)$ is bounded for large $t's$.

3.5 Neumann Boundary Conditions

Separation of variables, starting with $X(x)$ as previously found for the wave equation:

$$X(x) = C\cos\beta x + D\sin\beta x, \tag{3.5.1}$$

and Neumann boundary conditions,

$$u_x(0,t) = u_x(l,t) = 0. \tag{3.5.2}$$

With the left boundary condition $u_x(0,t) = 0$, we have

$$\begin{aligned} 0 &= X^{'}(0) \\ &= -C\beta\sin\beta 0 + D\beta\cos\beta 0 \\ &= 0 + D\beta(1), \end{aligned}$$

$\beta \neq 0$ for all cases, so $D = 0$. This gives $X(x) = C\cos\beta x$.

And for $u_x(l,t) = 0$,

$$0 = X^{'}(l) = -\beta C\sin\beta l,$$

we have $\beta_n = n\pi/l$, since $C = 0$ would lead to a trivial solution $X(x) \equiv 0$. Replacing β in $X(x) = C\cos\beta x$ gives

$$X_n(x) = C_n\cos\frac{n\pi x}{l},$$

this gives

$$u(x,t)_n = A_n e^{-(n\pi/l)^2kt}\cos\frac{n\pi x}{l} \tag{3.5.3}$$

For Dirichlet boundary conditions in last section, it could be shown that $\lambda \neq 0$; but for Neumann's boundary conditions, λ can be zero and $\lambda = 0$ adds the term, $\frac{1}{2}A_0$ to what we have; so now (3.5.3) looks like

$$u(x,t) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \cos\frac{n\pi x}{l}, \tag{3.5.4}$$

and zero is be included as an eigenvalue

$$\lambda = \beta^2 = \left(\frac{n\pi}{l}\right)^2 \qquad \text{for} \quad n = 0, 1, 2, 3, \ldots. \tag{3.5.5}$$

For Neumann BCs in (3.5.4), we have the cosine series in it, instead of the sine series of Dirichlet BCs. With Neumann's, now the initial conditions (at $t = 0$) looks like

$$u(x,0) = \phi(x) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}. \tag{3.5.6}$$

The cosine series. For Neumann's boundary conditions, $u_x(0,t) = u_x(l,t)$, we get for the diffusion equation's only initial condition $u(x,0) = \phi(x)$,

$$\begin{aligned}
\phi(x) &= u(x,0) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 k(0)} \cos\frac{n\pi x}{l} \\
&= \frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}.
\end{aligned}$$

Example 3.3. Solve the diffusion problem $u_t = ku_{xx}$ in $0 < x < l$, with the mixed boundary conditions $u(0,t) = u_x(l,t) = 0$.

Solution: Look for

$$\begin{aligned}
u(x,t) &= X(x)\,T(t), \\
\text{then} \quad X(x)\,T'(t) &= kX''(x)\,T(t), \\
\text{so} \quad \frac{X(x)\,T'(t)}{kX(x)\,T} &= \frac{kX''(x)\,T(t)}{kX(x)\,T}, \\
\text{gives} \quad \frac{T'(t)}{kT(x)} &= \frac{X''(x)}{X(x)} = -\lambda.
\end{aligned}$$

So $X''(x) + \lambda X(x) = 0$, then if $\lambda > 0$, we get

$$X(x) = C\cos\beta x + D\sin\beta x,$$

$$\text{where } \beta^2 = \lambda, \quad \text{and } \beta > 0.$$

The boundary condition $u(0,t) = 0$ implies $X(0)T(t) = 0$ for all $t \Longrightarrow X(0) = 0$, (we do not want $u \equiv 0$, which would be if $T(t) = 0$). So

$$\begin{aligned} 0 &= X(0) = C\cos\beta(0) + D\sin\beta(0) \\ &= C + D(0) \\ &\Longrightarrow C = 0, \end{aligned}$$

$$\text{hence,} \qquad X(x) = D\sin\beta x.$$

With $u_x(l,t) = 0$, we have $X'(l)T(t) = 0$ for all t, since $T(t) \neq 0$,

$$0 = X'(l) = \beta D\cos\beta(l),$$

but $\beta \neq 0$ and $D \neq 0$, then

$$\cos\beta l = 0$$

$$\Rightarrow \quad \beta_n l = \left(n + \frac{1}{2}\right)\pi \quad \text{so} \quad \beta_n = \frac{\left(n+\frac{1}{2}\right)\pi}{l},$$

$$\text{thus} \quad \lambda_n = \beta_n^2 = \left(\frac{\left(n+\frac{1}{2}\right)\pi}{l}\right)^2, \quad n = 0, 1, 2, \ldots.$$

$$\text{And} \qquad X_n(x) = D_n \sin\frac{\left(n+\frac{1}{2}\right)\pi}{l}x.$$

For T,

$$T'(t) + k\lambda T(t) = 0 \ \text{ gives } \ T(t) = Ae^{-k\lambda t}.$$

Note: there are no negative eigenvalues, for if $\lambda < 0$, we can write $\lambda = \gamma^2$ (where WLOG $\gamma > 0$),

then

$$\begin{aligned} X''(x) + \lambda X(x) &= 0, \text{ becomes } \; X''(x) - \gamma^2 X(x) = 0, \\ \text{so } \; X(x) &= A\cosh\gamma x + B\sinh\gamma x, \end{aligned}$$

and the boundaries give:

$$0 = X(0) = A \cdot 1 + B \cdot 0$$

$$\Rightarrow A = 0,$$

we get $\quad X(x) = B\sinh\gamma x.$

and $\quad 0 = X'(l) = B\gamma\cosh\gamma l \;$ where $B \neq 0, \; \gamma \neq 0, \; l \neq 0,$

$\Longrightarrow \;$ $\cosh\gamma l = 0,$ but this is not true, $\cosh x$ is never zero.

We have a contradiction; that is, there is no negative eigenvalue.

So $\qquad T_n(t) = A_n e^{-k\lambda_n t}$

and $\qquad u(x,t) = \sum_{n=0}^{\infty} B_n e^{-k\frac{\left(n+\frac{1}{2}\right)^2\pi^2}{l^2}t} \sin\left(\frac{\left(n+\frac{1}{2}\right)\pi}{l}x\right)$

where $\qquad B_n = A_n \cdot D_{n.}$

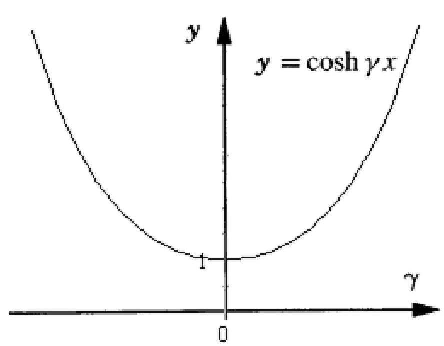

Figure 3: $\cosh\gamma l \neq 0$

3.6 The Robin boundary conditions

$$
\begin{aligned}
X' - a_0 X &= 0 \qquad \text{at } x = 0 \\
X' + a_l X &= 0 \qquad \text{at } x = l.
\end{aligned}
$$

where a_0 and a_l are constants. For $X(x) = C\cos\beta x + D\sin\beta x$ with the Robin boundary condition at $x = 0$ gives

$$
\begin{aligned}
0 &= X'(0) - a_0 X(0) \\
&= -\beta C \sin\beta 0 + \beta D \cos\beta 0 - a_0 (C\cos\beta 0 + D\sin\beta 0), \\
&= \beta D - a_0 (C) \\
\text{gives} \qquad D &= \frac{a_0 (C)}{\beta}
\end{aligned}
$$

similarly, at $x = l$

$$
\begin{aligned}
0 &= X'(l) + a_l X(l) \\
&= -\beta C \sin\beta l + \beta D \cos\beta l + a_l (C\cos\beta l + D\sin\beta l), \\
&= -\beta C \sin\beta l + \beta D \cos\beta l + a_l C\cos\beta l + a_l D\sin\beta l \\
&= -\beta C \sin\beta l + \beta \left(\frac{a_0 (C)}{\beta}\right)\cos\beta l + a_l C\cos\beta l \\
&\quad + a_l \left(\frac{a_0 (C)}{\beta}\right)\sin\beta l \\
&= -\beta C \sin\beta l + a_0 C\cos\beta l + a_l C\cos\beta l + \frac{a_l a_0 C}{\beta}\sin\beta l
\end{aligned}
$$

multiplying by β and factoring out C, we get

$$0 = -\beta^2 \sin\beta l + a_0\beta\cos\beta l + a_l\beta\cos\beta l + a_l a_0 \sin\beta l$$

$$\beta^2 \sin\beta l - a_l a_0 \sin\beta l = a_0\beta\cos\beta l + a_l\beta\cos\beta l$$

$$\left(\beta^2 - a_l a_0\right) \sin \beta l = (a_0 + a_l)\, \beta \cos \beta l$$

$$\left(\beta^2 - a_l a_0\right) \tan \beta l = (a_0 + a_l)\, \beta \cos \beta l,$$

"Any root $\beta > 0$ of this "algebraic" equation would give us an eiganvalue $\lambda = \beta^2$ " (Struass, p.94).

If $C \neq 0$, and $D = \dfrac{a_0\,(C)}{\beta}$, we get the corrsponding eigenfunction

$$X\,(x) = C \cos \beta x + \frac{C a_0}{\beta} \sin \beta x.$$

Solving for β is difficult in

$$\left(\beta^2 - a_l a_0\right) \tan \beta l = (a_0 + a_l)\, \beta \cos \beta l, \tag{3.6.1}$$

so will use graphing to analyze numerical values of β. Dividing (3.6.1) by $\cos \beta l$, we get the trigonometric identity $\dfrac{\sin \beta l}{\cos \beta l} = \tan \beta l$ and write (3.6.1) as

$$\tan \beta l = \frac{(a_0 + a_l)\,\beta}{\beta^2 - a_l a_0},$$

and find intersections of $\tan \beta l$ and $\dfrac{(a_0 + a_l)\,\beta}{\beta^2 - a_l a_0}$, as functions of $\beta > 0$.

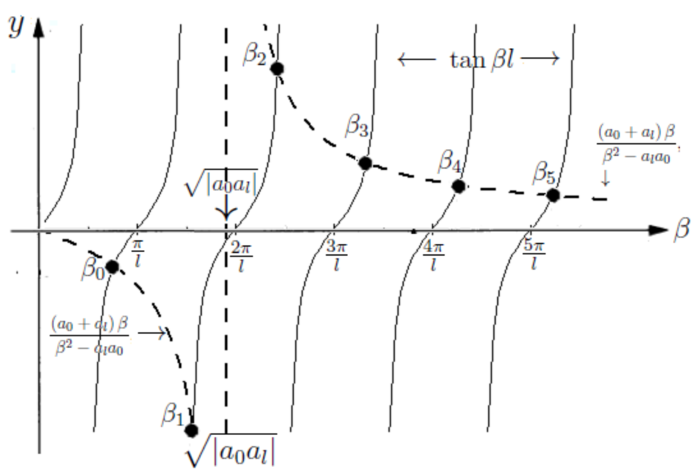

Figure 4: $a_0 > 0, a_l > 0$, eigenvalues as intersections

In figure (7), the eigenvalues are between zeros of $\tan \beta l$; this, and the intersections show that

$$n^2 \frac{\pi^2}{l^2} < \beta_n^2 = \lambda_n < (n+1)^2 \frac{\pi^2}{l^2} \qquad (n = 0, 1, 2, 3, \ldots). \tag{3.6.2}$$

Note that in figure (7), when $\cos \beta l = 0$, $\beta l = \frac{\pi}{2}$, and $\sin \frac{\pi}{2} = 1$, $(\beta^2 - a_l a_0) \sin \beta l = (a_0 + a_l)\, \beta \cos \beta l$, gives

$$\begin{aligned} 0 &= \left(\beta^2 - a_l a_0\right) \sin \beta l \\ &= \left(\beta^2 - a_l a_0\right) \\ \Longrightarrow \qquad \beta &= \sqrt{a_l a_0}, \end{aligned}$$

this is when "the tangent function and rational function 'intersect at infinity'" [4, p 94].

For the case of $a_0 < 0$ $a_l > 0$, and $a_0 + a_l > 0$, the maximum occurs at $\sqrt{|-a_0 a_l|}$; this can be shown by finding the critical point where $\frac{(-a_0 + a_l)\,\beta}{\beta^2 - (-a_0) a_l}$ reaches the maximum (see Figure (5)). We use the quotient rule for derivatives:

$$\left(\frac{(-a_0 + a_l)\,\beta}{\beta^2 + a_0 a_l} \right)' = \frac{(-a_0 + a_l)\,(\beta^2 + a_0 a_l) - (-a_0 + a_l)\,\beta\,(2\beta)}{\left(\beta^2 + a_0 a_l\right)^2}$$

and set the right side to zero, (implies the numerator is zero) and get

$$\begin{aligned} &(-a_0 + a_l)\left(\beta^2 + a_0 a_l\right) = (-a_0 + a_l)\, 2\beta^2 \\ \Rightarrow \quad &\beta^2 + a_0 a_l = 2\beta^2 \\ \Rightarrow \quad &\beta^2 = a_0 a_l \\ \Rightarrow \quad &\beta = \sqrt{|a_0 a_l|}\,. \end{aligned}$$

The intersections of $\tan\beta l = \dfrac{(-a_0+a_l)\,\beta}{\beta^2+a_l a_0}$ are shown in figure (5),

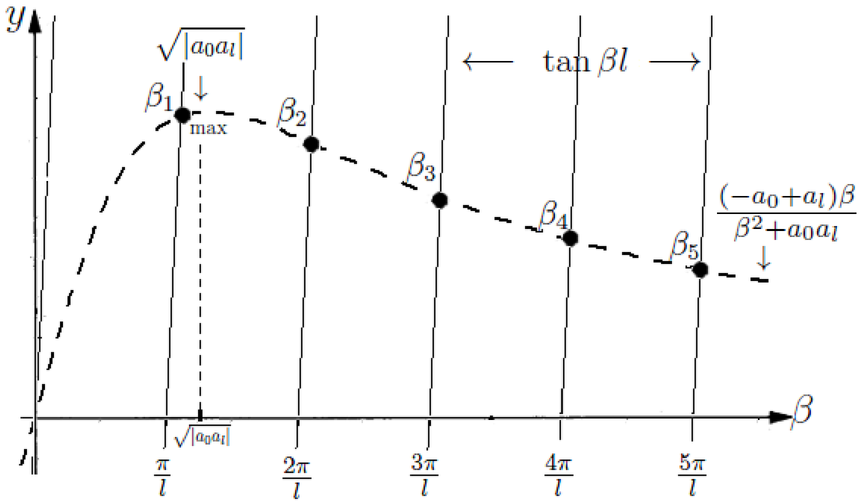

Figure 5: $a_0 < 0, a_l > 0$ and $a_0 + a_l > 0$, $\tan\beta l = \dfrac{(-a_0+a_l)\,\beta}{\beta^2+a_l a_0}$.

Example 3.4. Find the eigenvalues graphically for the boundary conditions $X(0) = 0, \quad X'(l) + aX(l) = 0, \quad$ for $\quad -X'' = \lambda X, \quad$ where $\lambda = \beta^2$. Assume that $a \neq 0$.

Solution: We have

$$\begin{aligned} X(x) &= C\cos\beta x + D\sin\beta x, \\ \text{and } \ 0 &= X(0) = C + D(0) \Rightarrow C = 0 \\ \text{so } X(x) &= D\sin\beta x. \end{aligned}$$

On the other hand, $X'(l) + aX(l) = 0$ gives:

$$\begin{aligned} 0 &= \beta D\cos\beta l + aD\sin\beta l \\ &\Longrightarrow -\beta D\cos\beta l = aD\sin\beta l \\ &\Longrightarrow -\frac{\beta}{a} = \frac{\sin\beta l}{\cos\beta l} \\ &\Longrightarrow -\frac{\beta}{a} = \tan\beta l. \end{aligned}$$

The intersection of functions $-\dfrac{\beta}{a}$ and $tan\beta l$, will give the eigenvalues:

Case 1: $a > 0$; so $y = -\frac{\beta}{a} < 0$: The discontinuities are at $\beta = \left(n - \frac{1}{2}\right)\pi/l$ and the roots of $\tan\beta l$ are $n\pi/2$, for $n = 1, 2, 3, \ldots$. We can see that

$$\frac{\left(n - \frac{1}{2}\right)\pi}{l} < \beta_n < \frac{n\pi}{l},$$

and also the graph shows that $\lim_{n\to\infty}\left(\beta_n - \frac{(n-\frac{1}{2})\pi}{l}\right) = 0.$

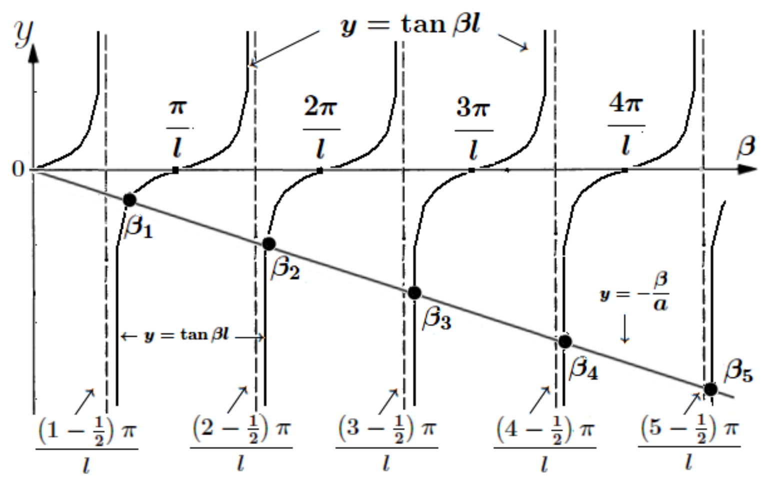

Figure 6: graph of $-\frac{\beta}{a} = \tan\beta l$, where $a > 0$

Case 2: $a < 0 \Longrightarrow y = -\frac{1}{a}\beta > 0$; so, the graph:

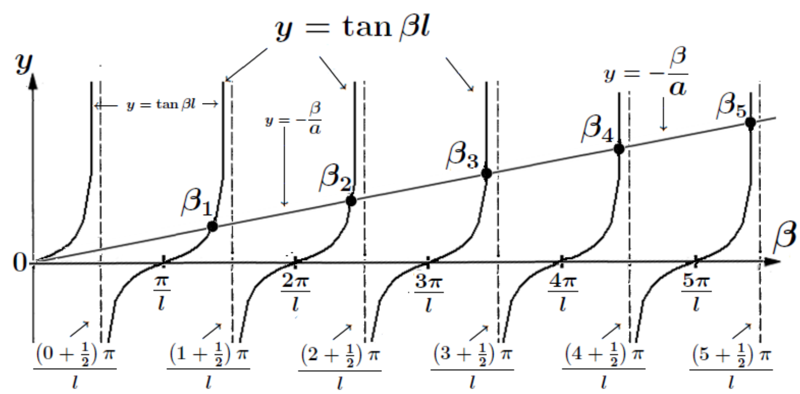

Figure 7: graph of $-\frac{\beta}{a} = \tan\beta l$, where $a < 0$

shows that $\frac{n\pi}{l} < \beta_n < \frac{\left(n+\frac{1}{2}\right)\pi}{l}$ and $\beta_n - \frac{\left(n+\frac{1}{2}\right)\pi}{l} \to 0_{\text{ as } n\to\infty}$. So, larger eigenvalues get closer to $\frac{\pi^2}{l^2}\left(n+\frac{1}{2}\right)^2$.

Example 3.5. We will use Newton's Method of iterations to comopute the first intersections, with $f(\beta) = (\tan \pi\beta)(\beta^2 - 3.61) - 3.8\beta$, $\beta_1 = 1$, $n = 1$, $L = \pi$ and $a_0 = a_l = 1.9$, where

$$\beta_{n+1} = \beta - \frac{f(\beta_n)}{f'(\beta_n)},$$

$$\beta_{n+1} = \beta - \frac{(\tan \pi\beta)(\beta^2 - 3.61) - 3.8\beta}{\pi(\tan^2 \pi\beta + 1)(\beta^2 - 3.61) + (\tan \pi\beta) 2\beta - 3.8}$$

$$\begin{aligned}
\beta_2 &= \beta_1 - \frac{(\tan \pi\beta_1)(\beta^2 - 3.61) - 3.8\beta_1}{\pi(\tan^2 \pi\beta_1 + 1)(\beta_1^2 - 3.61) + (\tan \pi\beta_1) 2\beta_1 - 3.8} = 0.683\,321\,638 \\
\beta_3 &= \beta_2 - \frac{(\tan \pi\beta_2)(\beta_2^2 - 3.61) - 3.8\beta_2}{\pi(\tan^2 \pi\beta_2 + 1)(\beta_2^2 - 3.61) + (\tan \pi\beta_2) 2\beta_2 - 3.8} = 0.740\,563\,595 \\
\beta_4 &= \beta_3 - \frac{(\tan \pi\beta_3)(\beta_3^2 - 3.61) - 3.8\beta_3}{\pi(\tan^2 \pi\beta_3 + 1)(\beta_3^2 - 3.61) + (\tan \pi\beta_3) 2\beta_3 - 3.8} = 0.757\,393\,701 \\
\beta_5 &= \beta_4 - \frac{(\tan \pi\beta_4)(\beta_4^2 - 3.61) - 3.8\beta_4}{\pi(\tan^2 \pi\beta_4 + 1)(\beta_4^2 - 3.61) + (\tan \pi\beta_4) 2\beta_4 - 3.8} = 0.758\,261\,759 \\
\beta_6 &= \beta_5 - \frac{(\tan \pi\beta_5)(\beta_5^2 - 3.61) - 3.8\beta_5}{\pi(\tan^2 \pi\beta_5 + 1)(\beta_5^2 - 3.61) + (\tan \pi\beta_5) 2\beta_5 - 3.8} = 0.758\,263\,778\,,
\end{aligned}$$

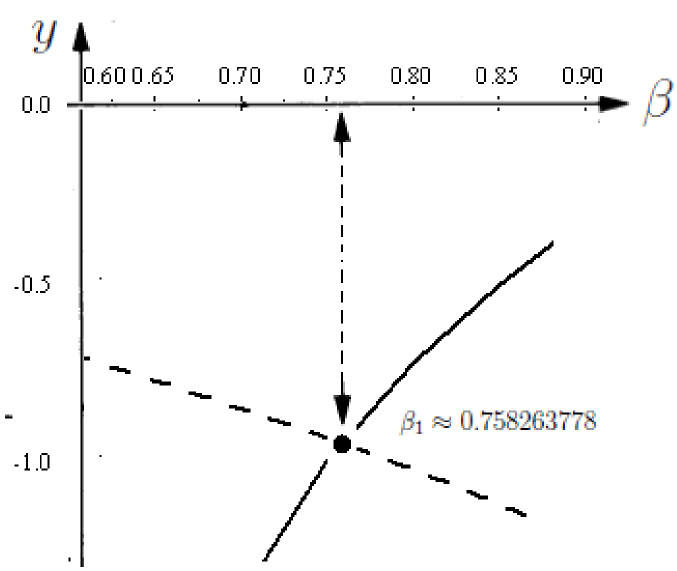

Figure 8: $a_0 = a_l = 1.9$, $l = \pi$, $\beta_1 \approx 0.758263778$

CHAPTER 4

Eigenvalues

Definition 4.1. *λ is an eigenvalue of a matrix A, if there is a vector v called eigenvector, such that*

$$Av = \lambda v \tag{4.0.1}$$

where $v \neq 0$.

Analogously, for the differential operator $-\dfrac{d^2}{dx^2}$ we have that

$$X''(x) = -\lambda X(x)$$
$$\Longrightarrow \qquad -\frac{d^2}{dx^2}X(x) = \lambda X(x),$$

where λ is an eigenvalue of $-\dfrac{d^2}{dx^2}$ for a nonzero function $X(x)$.

4.1 Eigenvalues: Dirichlet Boundary Conditions

The eigenvalues for *Dirichlet boundary* problems are all positive.

If $\lambda = 0$, then

$$\begin{aligned} X'' &= 0 \\ X'(x) &= D \\ X(x) &= Dx + C \\ \text{Derichlet BC} \qquad 0 &= X(0) = D(0) + C \\ \text{implies} \qquad C &= 0 \\ \text{so} \qquad X(x) &= Dx \end{aligned}$$

$$\text{for } X(l) = 0: \qquad 0 = X(l) = D(l)$$

$l \neq 0$, so $D = 0$, therefore zero is not an eigenvalue. For an eigenvalue, λ, the eigen-function X can not be zero.

Negative Eigenvalues?

If λ were negative, we would write it as $\lambda = -\gamma^2$, where $\gamma > 0$; then,

$$X'' = -\left(-\gamma^2\right) X = \gamma^2 X.$$

Suppose λ is negative, then we have two real solutions of the form e^{rx}, then by section (3.2), $X(x) = c_1 e^{\gamma x} + c_2 e^{-\gamma x}$ are solutions. For Dirichlet boundary conditions, $X(x) = 0$, we have

$$\begin{aligned}
0 &= X(0) = c_1 e^{\gamma(0)} + c_2 e^{-\gamma(0)} \\
&= c_1(1) + c_2(1) \\
\text{this gives} \qquad -c_1 &= c_2 \\
\text{so} \qquad X(x) &= c_1 e^{\gamma x} - c_1 e^{-\gamma x} \\
\text{at } X(l) = 0: \qquad 0 &= X(l) = c_1 e^{\gamma l} - c_1 e^{-\gamma l} \\
&= c_1 \left(e^{\gamma l} - e^{-\gamma l}\right) \\
&= e^{\gamma l} - e^{-\gamma l} \\
\text{we have} \qquad e^{\gamma l} &= e^{-\gamma l} \\
\text{taking ln}: \quad \ln e^{\gamma l} &= \ln e^{-\gamma l} \\
\text{we get} \qquad \gamma l &= -\gamma l
\end{aligned}$$

this is true if $\gamma l = 0$, a contradiction; then, the Dirichlet's boundary conditions does not allow negative eigenvalues.

4.2 Eigenvalues: Neumann Boundary Conditions

$$u_x(0,t) = u_x(l,t) = 0.$$

Zero Eigenvalues

If $\lambda = 0$, then $-X''(x) = 0$ implies $X'(x) = A$ and $X(x) = Ax + B$; A and B constants.

$$\begin{array}{lrcl} \text{left BC} & X'(0) & = & A \\ \text{gives} & A & = & 0 \\ \text{then} & X(x) & = & B \\ \text{and} & X'(x) & = & 0. \\ \text{righ BC} & X'(l) & = & 0 \end{array}$$

Then $X(x) = B$ is not zero; therefore, for the Neumann's boundary conditions we have a zero eigenvalue when $-X'' = 0$.

Negative Eigenvalues?

Let λ be a negative eigenvalue and write $\lambda = -\gamma^2$; by the characteristic polynomial method

$$X'' = -\left(-\gamma^2\right) X = \gamma^2 X$$

$$\text{implies} \qquad r = \pm\gamma.$$

We have two real solutions of the form e^{rx}, so the solution is

$$X(x) = c_1 e^{\gamma x} + c_2 e^{-\gamma x}.$$

$$\begin{aligned}
\text{at } X'(0): \qquad 0 &= X'(0) = \gamma c_1 e^{\gamma(0)} - \gamma c_2 e^{-\gamma(0)} \\
\frac{0}{\gamma} &= \frac{\gamma}{\gamma}(c_1 - c_2) \\
\text{gives} \qquad -c_1 &= c_2 \\
\text{we have} \qquad X(x) &= c_1 e^{\gamma x} - c_1 e^{-\gamma x} \\
\text{at } X'(l): \qquad 0 &= X'(l) = c_1 \gamma \left(e^{\gamma l} + e^{-\gamma l}\right) \\
\text{then} \qquad e^{\gamma l} &= -e^{-\gamma l} \\
\text{this gives} \qquad e^{2\gamma l} &= -1.
\end{aligned}$$

We have a contradiction because e^x is never -1 for $-\infty < x < \infty$. So we conclude $\lambda = -\gamma^2$ is not a negative eigenvalue when Neumann's boundary conditions are applied to the solution $u(x,t) = X(x)T(t)$.

4.3 Eigenvalues: The Robin's Boundary Conditions

$$\text{For} \qquad -X'' = \lambda X$$

$$\text{and} \qquad X'(0) - a_0 X(0) = 0, \quad X'(l) + a_0 X(l) = 0.$$

Zero Eigenvalues?

If $\lambda = 0$, then $X(x) = C + Dx$, and $X'(x) = D$, so

$$\begin{aligned}
X'(x) - a_0 X(x) &= D - a_0(C + Dx) \\
\text{and} \qquad 0 &= X'(0) - a_0 X(0) \\
\text{implies} \qquad 0 &= D - a_0(C + D(0)) \\
&= D - a_0 C
\end{aligned}$$

$$
\begin{aligned}
\text{then} \qquad D &= a_0 C \\
\text{so} \qquad X(x) &= C + (a_0 C)\, x \\
\text{and} \qquad 0 &= X'(l) + a_l X(l) \\
\text{gives} \qquad &= a_0 C + a_l (C + a_0 C l) \\
&= a_0 C + a_l C + a_l a_0 C l \\
\text{then } \lambda = 0 \text{ if and only if } \ 0 &= C (a_0 + a_l + a_l a_0 l) \\
\text{that is, if } C = 0 \quad \text{or} \qquad 0 &= a_0 + a_l + a_l a_0 l \\
\text{if } C = 0 \text{ then } D = 0, \Rightarrow X &= 0 \quad \text{not an eigen function} \\
\text{so} \qquad 0 &= a_0 + a_l + a_l a_0 l \\
\text{then} \qquad -a_l a_0 l &= a_0 + a_l, \quad \text{if and only if } \lambda = 0.
\end{aligned}
$$

Negative Eigenvalues?

Let $\lambda = -\gamma^2 < 0$, then λ is negative. This gives $-X'' = -\gamma^2 X$, by the characteristic polynomial method we have

$$X(x) = c_1 e^{\gamma x} + c_2 e^{-\gamma x} = A \cosh \gamma x + B \sinh \gamma x,$$

and $X'(0) - a_0 X(0) = 0$ gives

$$
\begin{aligned}
0 &= A\gamma \sinh 0\gamma + B\gamma \cosh 0\gamma - a_0 (A \cosh \gamma 0 + B \sinh \gamma 0) \\
&= 0 + B\gamma - a_0 A - 0 \\
\text{we get} \qquad B &= \frac{a_0 A}{\gamma} \\
\text{so} \qquad X(x) &= A \cosh \gamma x + \frac{a_0 A}{\gamma} \sinh \gamma x \\
\text{Then} \qquad 0 &= X'(l) + a_l X(l)
\end{aligned}
$$

$$
\begin{aligned}
&= A\gamma \sinh \gamma l + a_0 A cosh \gamma l + a_l \left(A \cosh \gamma x + \frac{a_0 A}{\gamma} \sinh \gamma l \right) \\
\frac{\gamma}{A} 0 &= \gamma^2 \sinh \gamma l + a_0 \gamma \cosh \gamma l + a_l \gamma \cosh \gamma l + a_l a_0 \sinh \gamma x \\
\left(\gamma^2 + a_l a_0 \right) \sinh \gamma l &= -\left(a_0 + a_l \right) \gamma \cosh \gamma l \\
\left(\gamma^2 + a_l a_0 \right) \frac{\sinh \gamma l}{\cosh \gamma l} &= -\left(a_0 + a_l \right) \gamma \\
\tanh \gamma x &= -\frac{\left(a_0 + a_l \right) \gamma}{\gamma^2 + a_l a_0}
\end{aligned}
$$

We graph both sides; if there is an intersection, then we will have a negative eigenvalue(s). With a_0 and a_l, both positive:

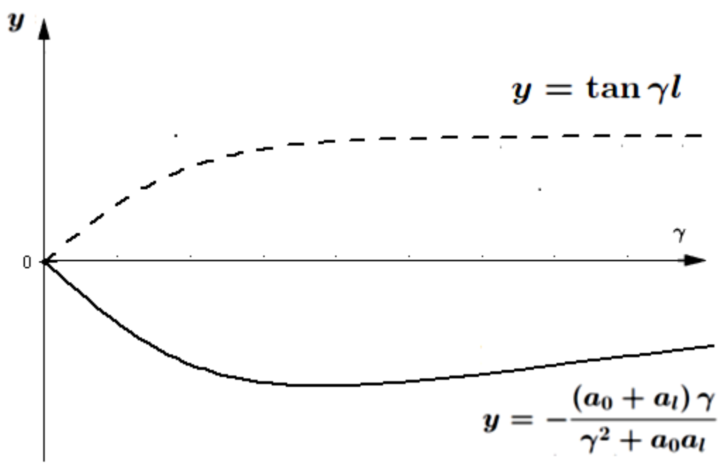

Figure 9: for $a_0 > 0,\ a_l > 0$, no negative eigenvalue

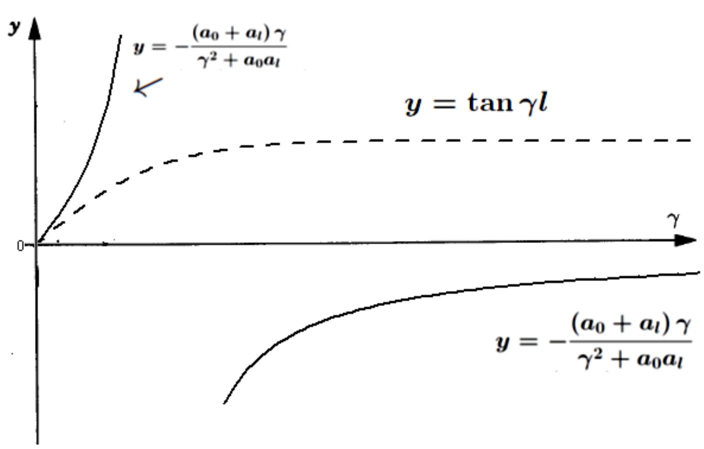

Figure 10: for $a_0 < 0,\ a_l > 0$, $a_0 + a_l > 0$, no negative eigenvalue.

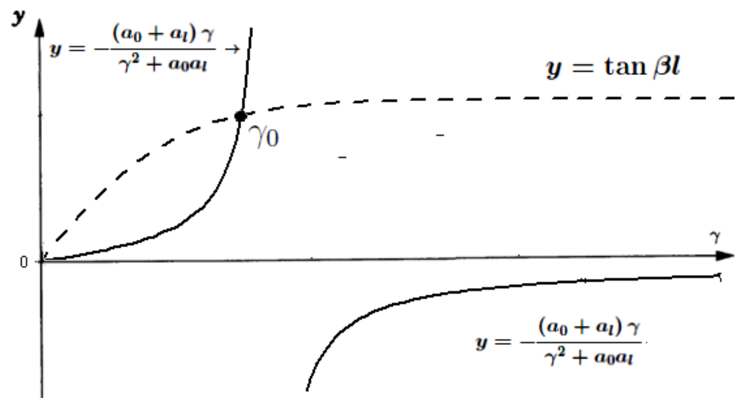

Figure 11: for $a_0 < 0,\ a_0 + a_1 < -a_0 a_l l$, one negative eigenvalue where the functions intersect.

CHAPTER 5

Coefficients

5.1 Coefficients in the Case of Dirichlet Boundary Conditions

To find coefficient C_n in (3.3.6) we start by setting (3.3.6) to the initial condition $u(x,0)$

$$\begin{aligned} \phi(x) &= u(x,0) = \sum_{n=1}^{\infty}\left(C_n\cos\frac{n\pi c(0)}{l} + D_n\sin\frac{n\pi c(0)}{l}\right)\sin\frac{n\pi x}{l} \\ &= \sum_{n=1}^{\infty}(C_n+0)\sin\frac{n\pi x}{l} \\ &= \sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{l} \end{aligned}$$

This gives $\phi(x) = \sum_{n=1}^{\infty}C_n\sin\frac{n\pi x}{l}$, then multiply both sides by $\sin\frac{m\pi x}{l}$, and integrate from 0 to l term by term

$$\int_0^l \phi(x)\sin\frac{m\pi x}{l}dx = \sum_{n=1}^{\infty}C_n\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx$$

To compute this we use the trigonometric identities

$$\begin{aligned} \cos(a-b) &= \cos a\cos b + \sin a\sin b \\ \text{and} \qquad \cos(a+b) &= \cos a\cos b - \sin a\sin b \end{aligned}$$

combine cosines

$$\begin{aligned} \cos(a-b)-\cos(a+b) &= \cos a\cos b + \sin a\sin b - (\cos a\cos b - \sin a\sin b) \\ &= \cos a\cos b + \sin a\sin b - \cos a\cos b + \sin a\sin b \\ &= 2\sin a\sin b \\ \text{so} \qquad 2\sin a\sin b &= \cos(a-b)-\cos(a+b) \end{aligned}$$

$$\text{and} \qquad \sin a \sin b = \frac{1}{2}\cos(a-b) - \frac{1}{2}\cos(a+b)$$

Now we find the coefficient; first, we integrate for $m \neq n$; and we will need $\theta_1 = \dfrac{n-m}{l}\pi x$, $\dfrac{l}{n-m}d\theta_1 = dx$, similarly, $\theta_2 = \dfrac{n+m}{l}\pi x$, $\dfrac{l}{n+m}d\theta_2 = dx$

$$\begin{aligned}
&\int_0^l \frac{1}{2}\cos\left(\frac{n\pi x}{l} - \frac{m\pi x}{l}\right) - \frac{1}{2}\cos\left(\frac{n\pi x}{l} + \frac{m\pi x}{l}\right) dx \\
&= \int_0^l \frac{1}{2}\cos\left(\frac{n-m}{l}\pi x\right) - \frac{1}{2}\cos\left(\frac{n+m}{l}\pi x\right) dx \\
&= \int_0^l \frac{1}{2}\cos\left(\frac{n-m}{l}\pi x\right) dx - \int_0^l \frac{1}{2}\cos\left(\frac{n+m}{l}\pi x\right) dx \\
&= \int_0^l \frac{1}{2}\cos(\theta_1)\frac{l}{(n-m)\pi}d\theta_1 - \int_0^l \frac{1}{2}\cos(\theta_2)\frac{l}{(n+m)\pi}d\theta_2
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2}\frac{l}{(n-m)\pi}\sin\left(\frac{n-m}{l}\pi x\right)\Bigg|_0^l - \frac{1}{2}\frac{l}{(n+m)\pi}\sin\left(\frac{n+m}{l}\pi x\right)\Bigg|_0^l \\
&= \frac{1}{2}\frac{l}{(n-m)\pi}(\sin((n-m)\pi) - (\sin 0)) - \frac{1}{2}\frac{l}{(n+m)\pi}(\sin((n+m)\pi) - (\sin 0)) \\
&= 0
\end{aligned}$$

Integrate when m is fixed and $m = n$, we will need $\theta_3 = \dfrac{n\pi x}{l}$ and $d\theta_3 = \dfrac{n\pi}{l}dx$, so that $\dfrac{l}{n\pi}d\theta_3 = dx$ is used in substitution at a step of integration; also, we make use of the trigonometric identity $\sin^2 x = \dfrac{1}{2} - \dfrac{1}{2}cos2x$

$$\begin{aligned}
&\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx \\
&= \int_0^l \sin^2\frac{n\pi x}{l}dx \\
&= \int_0^l \frac{1}{2} - \frac{1}{2}\int_0^l \cos\frac{2n\pi x}{l}dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^l \frac{1}{2}dx - \frac{l}{n\pi}\int_0^l \frac{1}{2}\cos\theta_3 d\theta_3 \\
&= \frac{1}{2}l - \frac{1}{2}0 - \frac{1}{2}\frac{l}{2n\pi}\left(\sin 2n\pi - \sin 0\right) \\
&= \frac{1}{2}l - 0 - \frac{1}{2}\frac{l}{2n\pi}\left(0 - 0\right) \\
&= \frac{1}{2}l
\end{aligned}
$$

So $\displaystyle\int_0^l \sin\frac{n\pi x}{l}\sin\frac{n\pi x}{l}dx = \begin{cases} 0, & \text{if } m \neq n, \\ \frac{1}{2}l, & \text{if } m = n. \end{cases}$

$$
\begin{aligned}
\text{And} \quad & \int_0^l \phi(x)\sin\frac{m\pi x}{l}dx \\
&= \sum_{n=1}^{\infty} C_n \int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx \\
&= C_n \sum_{n=1}^{\infty} \int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx \\
&= C_1 \int_0^l \sin\frac{(1)\pi x}{l}\sin\frac{m\pi x}{l}dx \\
& \quad + C_2 \int_0^l \sin\frac{2\pi x}{l}\sin\frac{m\pi x}{l}dx \\
& \quad + \cdots + C_m \int_0^l \sin\frac{m\pi x}{l}\sin\frac{m\pi x}{l}dx \\
& \quad + C_{m+1} \int_0^l \sin\frac{n\pi x}{l}\sin\frac{(m+1)\pi x}{l}dx + \cdots \\
& \quad = 0 + 0 + \cdots + C_m \cdot \frac{1}{2} + 0 + 0 + \cdots
\end{aligned}
$$

This gives $\displaystyle\int_0^l \phi(x)\sin\frac{m\pi x}{l}dx \overset{m=m}{=} C_m \cdot \frac{l}{2}, \quad$ or $\quad \displaystyle C_m = \frac{2}{l}, \int_0^l \phi(x)\sin\frac{m\pi x}{l}dx$

this is the Fourier coefficient formula for C_n. The other initial condition, $u_t(x,0)$, for the wave gives

$$
u_t(x,t) = \sum_{n=1}^{\infty}\left(-\frac{n\pi c}{l}C_n\sin\frac{n\pi ct}{l} + \frac{n\pi c}{l}D_n\cos\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}
$$

$$\begin{aligned}
\text{and} \qquad u_t(x,0) &= \sum_{n=1}^{\infty}\left(-\frac{n\pi c}{l}C_n\sin 0+\frac{n\pi c}{l}D_n\cos 0\right)\sin\frac{n\pi x}{l} \\
&= \sum_{n=1}^{\infty}\frac{n\pi c}{l}D_n\sin\frac{n\pi x}{l} \\
\text{so} \qquad \psi(x) &= \sum_{n=1}^{\infty}\frac{n\pi c}{l}D_n\sin\frac{n\pi x}{l} \\
\text{and we get} \qquad \frac{n\pi c}{l}D_n &= \frac{2}{l}\int_0^l \psi(x)\sin\frac{m\pi x}{l}dx \quad \text{by the same process we got } C_n.
\end{aligned}$$

5.2 Coefficients in the Case of Neumann Boundary Conditions

To find the constant coefficient A_n, we start the series at $n=0$, (to include A_0) and follow the same steps as we did in finding coefficients for Dirichlet's boundary conditions; but now we have $\cos\frac{n\pi x}{l}$, instead of $\sin\frac{n\pi x}{l}$, so we add the cosine difference and cosine sum to get the needed trigonometric identity:

$$\begin{aligned}
\text{then} \qquad \cos(a-b)+\cos(a+b) &= \cos a\cos b+\sin a\sin b+(\cos a\cos b-\sin a\sin b) \\
&= \cos a\cos b+\sin a\sin b+\cos a\cos b-\sin a\sin b \\
&= 2\cos a\cos b \\
\text{so} \qquad 2\cos a\cos b &= \cos(a-b)+\cos(a+b) \\
\text{and} \qquad \cos a\cos b &= \frac{1}{2}\cos(a-b)+\frac{1}{2}\cos(a+b)
\end{aligned}$$

Again, to integrate term by term, we set $m \neq n$, and we will get 0, because multiplying π by any integer in $\sin\dfrac{m\pi l}{l}=\sin\pi m=0$, and $\sin 0=0$, when computing the integral

$$\int_0^l \frac{1}{2}\cos\left(\frac{n\pi x}{l}-\frac{m\pi x}{l}\right)+\frac{1}{2}\cos\left(\frac{n\pi x}{l}+\frac{m\pi x}{l}\right)dx=0$$

and for the other integral, as we did before, we fix m and set it equal to n; plus we use the identity $\cos^2 x = \frac{1}{2} + \frac{\cos 2x}{2}$, where $\int_0^l \frac{1}{2} + \frac{\cos 2x}{2} dx = \frac{1}{2} l + 0$,

$$\begin{aligned}
\int_0^l \sin \frac{n\pi x}{l} \sin \frac{m\pi x}{l} dx &= \int_0^l \cos^2 \frac{n\pi x}{l} dx \\
&= \frac{1}{2} l \\
\text{so we get} \qquad A_m &\equiv \frac{2}{l} \int_0^l \phi(x) \cos \frac{m\pi x}{l} dx \quad \text{where } m = 0, 1, 2, \ldots.
\end{aligned}$$

For $m = 0, 1, 2, 3, \ldots$, this is coefficient A_m's formula for the cosine series

$$\frac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l}.$$

Example 5.1. Solve the Diffusion problem with Dirichlet boundary conditions and initial conditions.

$$u_t = k u_{xx}$$

$$u(0, t) = 0 = u(l, t)$$

$$u(x, 0) = 1 = \phi(x)$$

We know that

$$u(x, t) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 kt} \sin \frac{n\pi x}{l}.$$

the initial condition gives

$$\begin{aligned}
1 &= u(x, 0) = \sum_{n=1}^{\infty} A_n e^{-(n\pi/l)^2 k(0)} \sin \frac{n\pi x}{l} \\
&= \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{l}
\end{aligned}$$

and $A_m = \frac{2}{l} \int_0^l \phi(x) \sin \frac{m\pi x}{l} dx$, is the formula to find A.

Letting $\theta = \dfrac{m\pi x}{l}$, $\quad \theta d = \dfrac{m\pi}{l} dx$, $\quad$ then $\dfrac{l}{m\pi}\ \theta d = dx$,

$$\begin{aligned} A_m &= \frac{2}{l}\int_0^l \sin\theta \frac{l}{m\pi}\theta d \\ &= -\frac{2}{m\pi}\cos\theta\Big|_0^l \\ &= -\frac{2}{m\pi}\cos\frac{m\pi x}{l}\Big|_0^l \\ &= -\frac{2}{m\pi}\cos m\pi + \frac{2}{m\pi}, \end{aligned}$$

we have $\cos m\pi = (-1)^m$, and

$$-\frac{2}{m\pi}(-1)^m + \frac{2}{m\pi} = \begin{cases} \dfrac{4}{m\pi} & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even,} \end{cases}$$

so we get

$$\begin{aligned} 1 = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l} &= \sum_{n=1}^{\infty}\frac{4}{m\pi}\sin\frac{n\pi x}{l} \\ &= \frac{4}{m\pi}\left(\sin\frac{\pi x}{l} + \sin\frac{2\pi x}{l} + \sin\frac{3\pi x}{l} + \cdots\right). \end{aligned}$$

is an infinite series expansion.

The method of separation of variables is very helpful for solving linear PDEs of 2^{nd} order. Hwoever, it has its limitations. For problems with non-constant coefficients or for those with non-symmetric boundary conditions, the method will not work. Other methods would have to be explored in those cases.

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